What's in a Name?

Categorical Limits and the Utility of thinking about them as limits.

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Outline of a Standard Talk:

One shold speak about

- Things a general (mathematical) audience will understand

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- Things only specialists in your sub-discipline will understand
- Things only you understand
- Things no one understands

Most importantly, though

Be Original

So,

in my talk we will begin with

Things only specialists will understand

and move to

- Things a general audience would understand

and move to

- Things a general audience would understand
- Things ony I understand

and move to

- Things a general audience would understand
- Things ony I understand
- Things no one understands

Things only specialists may understand:

We “recall” the definitions of

1. Category
2. Limit
3. Final

Category

A $Category$ **C** is a collection of objects which we will denote $A, B, C, ...$ and, for each pair of objects, a set of arrows $Hom(A, B)$ satisfying the following:

For all objects A, B, C of **C**

- There exists a function $\circ : Hom(A, B) \times Hom(B, C) \longrightarrow Hom(A, C)$ $f \times g \mapsto g \circ f$, which is associative (i.e $h \circ (g \circ f) = (h \circ g) \circ f$).

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- There exists an arrow $1_A : A \longrightarrow A$ so that $f \circ 1_A = f$ and $1_A \circ g = g$.

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One advantage

of category theory is that it affords us the chance to "see" the relationships between objects and arrows.

Such a depiction is referred to as a *diagram*. They look like this:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ C & \xrightarrow[c]{} & D \end{array}$$

If

it is the case that in the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle b} \\ C & \xrightarrow[c]{} & D \end{array}$$

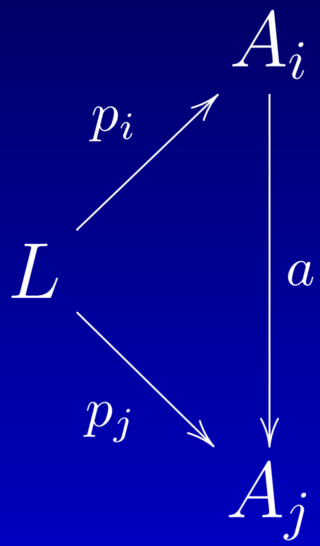
$b \circ f = c \circ g$, then the diagram is said to $commute$.

Limits

Suppose we have a diagram of objects $\{A_i\}_{i \in I}$ and arrows between them. Then, depending on the category, this diagram may have what we refer to as a *limit*

Definition

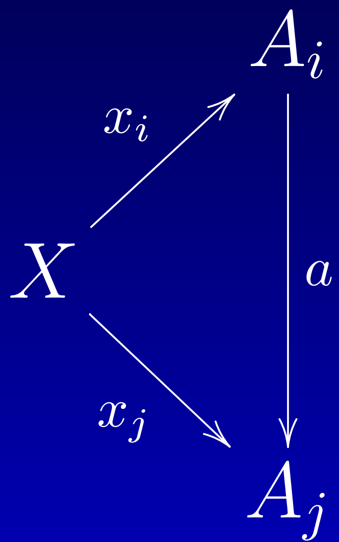
A *limit* of a diagram as in the last slide is an object L and arrows $\{p_i : L \longrightarrow A_i\}_{i \in I}$ so that for all $i, j \in I$ and for all arrows in the original diagram $a : A_i \longrightarrow A_j$, the diagram



commutes. In addition,

if

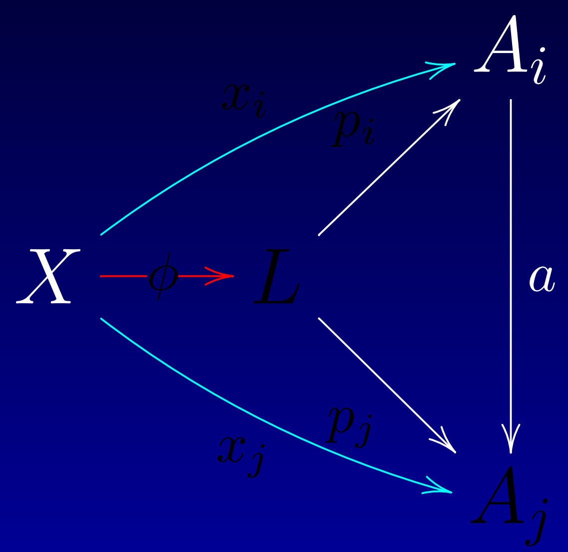
X and $\{x_i : X \to A_i\}$ is another object and collection of arrows (resp.) with the property that for all $i, j \in I$ and a, the diagram



also commutes, then

there exists a unique

ϕ so that



commutes.

Lastly

or, er..., $finally$, a subset $K \subset I$ of the index of the collection of objects $\{A_i\}_{i \in I}$ is said to be $final$ if for all $i \in I$ there exists $k \in K$ so that there exists an arrow $z : A_k \longrightarrow A_i$.

Q1:

Why is this important, and

Q2:

What does this have to do with the limits we learned about in (pre-) Calculus?

A1:

It turns out that the limit over a final subcollection is isomorphic to the limit over the whole collection.

A2:

falls into the category of

Things That a General Audience Knows

Suppose that $\{a_n\}$ is a sequence and that $\lim a_n > 0$. Then, we know that $\{a_n\}$ must eventually all be positive as well.

What - I thought - only I knew

but it turns out anyone can figure out: By using the fact that limits over final subsets are isomorphic to limits over the whole set, and a little logic we can prove something similar about categorical limits.

Theorem

Let P be a property of an object in a category $\mathbf{C}$. Suppose that the limit of objects $\{A_i\}_{i \in I}$ which do not have property P also does not have property P. Then there exists $i \in I$ so that if $j \in I$ and there exists $a : A_j \rightarrow A_i$, A_j has property P.

Proof:

Let $J \subset I$ be defined by all objects A_i which do not have property P. If J were final, the limit L would not have property P. The conclusion comes from negating the definition of $final$.

Corollary

Say the indexing set is totally ordered(for example, $\mathbb{N}$ with the usual ordering). Then this says that if the limit has some property there exists $n \in \mathbb{N}$ so that $n \leq m$ implies that A_m has this property.

Like

the property P : “is positive.”

Applications to Research

A Property

which satisfies the hypothesis - under certain conditions -of the theorem is “flatness”. Flatness plays an integral role in the problem of Local Weak Simultaneous Resolution (LWSR).

LWSR

Suppose $V \longrightarrow W$ is a map of varieties. Then there exists blow-ups $\tilde{V} \longrightarrow V$ and $\tilde{W} \longrightarrow W$ so that the diagram

$$\begin{array}{ccc} \tilde{V} & \longrightarrow & V \\ \downarrow & & \downarrow \\ \tilde{W} & \longrightarrow & W \end{array}$$

commutes and the canonical map $\tilde{V} \longrightarrow \tilde{W}$ is flat.

A proof of LSWR would complete the proof of desingularization of 3 dimensional varieties and give a big leg up on the desingularization of those in higher dimension.

How

is the theorem we discussed relevant to this problem?

The limit of the diagram of all possible blow-ups as above is the induced map between function fields. Such a map is *always* flat.

Things No One Knows

What are necessary and sufficient conditions for the the limit of non-flat objects to be non-flat. We know that all maps being finitely presented is sufficient.

Unfortunately the canonically induced map $\tilde{V} \to \tilde{W}$ is only rarely finitely presented.