Quiz

Define the term Greatest Lower Bound

Last Time:

1. Upper and Lower Bounds
2. L.U.B’s and G.L.B.’s
3. Directed Graphs
4. Duality
5. Top and Bottom Elements

This Time:

1. Maximal and Minimal Elements
2. Lattices
3. Foundations or “The Barber of Seville”

Maximal and Minimal Elements

Like

Top and $Bottom$

$UpperBound$ and $LowerBound$

$LeastUpperBound$ and $GreatestLowerBound$

The notions

$Maximal$ and $Minimal$
are dual.

Definition of Maximal:

An element m of a partially ordered set $(X, \leq)$ is called $maximal$ if for all $x \neq m \in X$, $m \nleq x$.

Definition of Minimal:

An element m of a partially ordered set $(X, \leq)$ is called $minimal$ if for all $x \neq m \in X$, $x \not\leq m$.

Maximal vs. Top Elements

Fact: Every top element is also a maximal element

Proof. Exercise. □

Lattices

Definition 1. *A partially ordered set in which every pair of elements has a least upper bound is called a lattice*

Example

Let X be a set and $\mathcal{P}(X)$ its power set. Then $(\mathcal{P}(X), \subseteq)$ is a lattice.

Proof

We have already seen that $(\mathcal{P}(X), \subseteq)$ is a partiall ordered set, so we have only left to prove that any two subsets $A \subseteq X$ and $B \subseteq X$ possess both a greatest lower and least upper bound. Let's handle the least upper bound first:

...

Clearly, the relations $A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$ hold. Suppose now that both $A \subseteq C$ and $B \subseteq C$ also hold for some $C \subseteq X$. Let $x \in A \bigcup B$. Then, by definition of $\bigcup$, $x \in A$.

Since $A \subseteq C$, $x \in C$. Similarly $x \in A \bigcup B$ implies that $x \in B$ and so $x \in C$ since $B \subseteq C$. Thus, $A \bigcup B \subseteq C$. Since C was an arbitrary set containing both A and B we can conclude that every set containing A and B also contains $A \bigcup B$.

Thus, by definition, $A \bigcup B$ is the least upper bound of A and B in $(\mathcal{P}, \subseteq)$.

What's Left?

Similarly, $A \bigcap B$ is the greatest lower bound of A and B
The proof of this is...

An Exercise

In coming sections we will meet other important lattices. For now, let us establish a notational convention which will prove useful then. If $\{x_i\}_{i=1}^n$ are n elements of a partially ordered set X, then we will denote their greatest lower bound $\wedge_{i=1}^n x_i$ and least upper bound $\vee_{i=1}^n x_i$/

Foundations

or... “The troubles of the Barber of Seville”

C

ategory theory has as its main concern collections of mathematical objects grouped by species and maps which preserve the essential structure which defines that particular species.

Thus, we will often use phrases such as “the collection of all X” where X stands for one of these types of mathematical objects.

The Problem:

comes from using phrases such as “the collection of all . . . ”

Russel's Paradox:

Let $T := \{S | S \notin S$ where S is any set. The question: Is $T \in T$?

2 possibilities:

1. $T \in T$
2. $T \notin T$

What's going on here?

If the barber of Seville shaves everyone who does not shave himself, who shaves the barber?

Answer:

His Wife!

Really,

In both cases there is no limit to which sets we might consider, no limit to the men who may or may not shave themselves

The Remedy:

Define a “Universe of discourse” all of whose elements are sets but, which, itself, is not.

Henceforth,

We will tacitly assume that all reasoning takes place within such a universe.

Next Time:

1. Read pages 11-14 in the text
2. Memorize the definition of any terms highlighted in $green$ in those pages for the quiz on Friday.