Quiz

Suppose A and B are subobjects of an object G. Define $A \leq B$ in the partially ordered set of subobjects of G.

Last Time:

We started the proof that the partially ordered set of subobjects in $\mathbf{AbG}$ is a lattice. This meant

- Any two - and thus any finite number of - subobjects has a least upper bound

Last Time:

We started the proof that the partially ordered set of subobjects in $\mathbf{AbG}$ is a lattice. This meant

- Any two - and thus any finite number of - subobjects has a least upper bound
- Any two - and thus any finite number of - subobjects has a greatest lower bound. This gets only a (since we proved only that AB is an upper bound for A and B, not the *least* upper bound.

Last Time:

We started the proof that the partially ordered set of subobjects in $\mathbf{AbG}$ is a lattice. This meant

- Any two - and thus any finite number of - subobjects has a least upper bound
- Any two - and thus any finite number of - subobjects has a greatest lower bound. This gets only a (since we proved only that AB is an upper bound for A and B, not the *least* upper bound.

This Time:

We complete the proof that what we have denoted AB is the least upper bound of A and B

Completion of the Proof

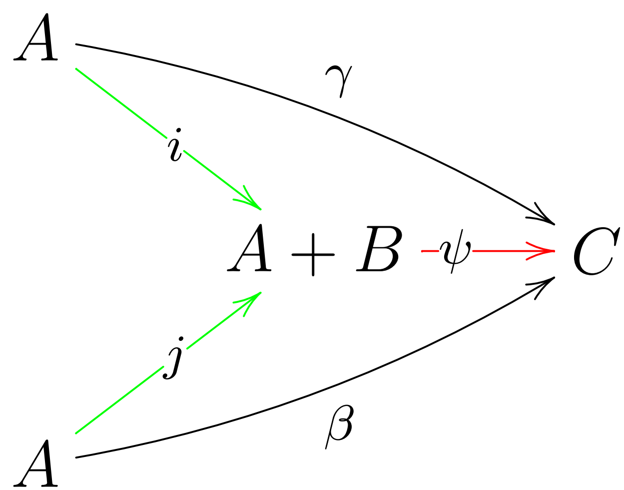
Suppose, now, that C is an upper bound for the set $\{< A, a >; < B, b >\}$. Then we have the commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & C \\ & \searrow^{a} & \downarrow{c} \\ & & G \end{array}$$

and,

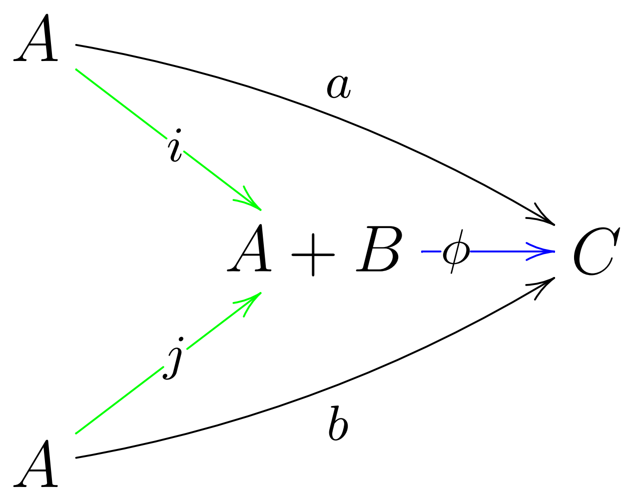
$$\begin{array}{ccc} B & \xrightarrow{\beta} & C \\ & \searrow^{b} & \downarrow{c} \\ & & G \end{array}$$

where β and γ are monic. This implies that there exists a unique ψ so that



commutes; where the green diagram is the sum diagram for A and B.

Let’s reset the table so to speak: Let



be the sum diagram (in green again) and ϕ the unique arrow making the diagram commute. Now,

$$\psi \circ i = \gamma \quad \text{thus,} \tag{1}$$

$$c \circ \psi \circ i = c \circ \gamma \quad \text{thus,} \tag{2}$$

$$= a \tag{3}$$

Similarly,

$$\psi \circ j = \beta \quad \text{thus,} \tag{4}$$

$$c \circ \psi \circ j = c \circ \beta \quad \text{thus,} \tag{5}$$

$$= b \tag{6}$$

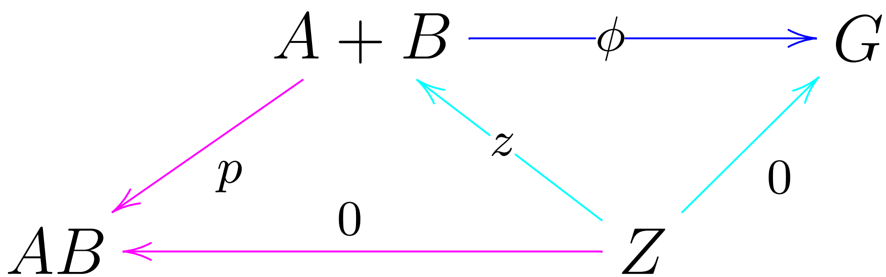
However, ϕ is the unique arrow so that

$$\phi \circ i = a \quad \text{and} \tag{7}$$

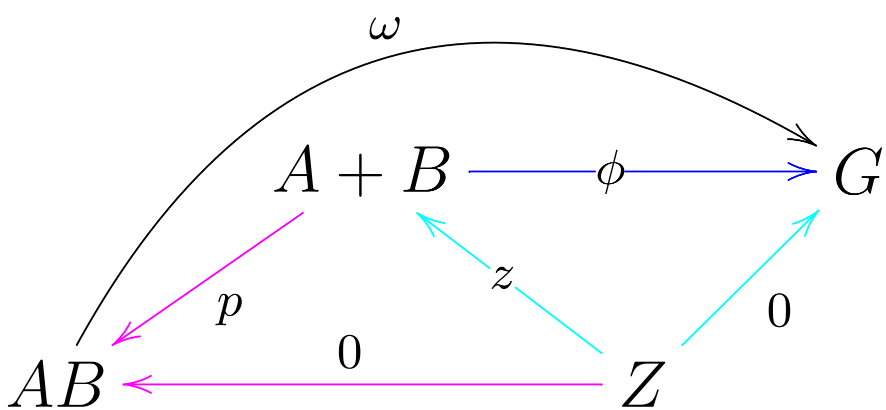
$$\phi \circ j = b \tag{8}$$

Thus, it must be that $c\psi = \phi$.

Now,let



be the coimage diagram for ϕ. Then, as we have seen, there exists a unique monic arrow ω so that



commutes

Now, we can see that

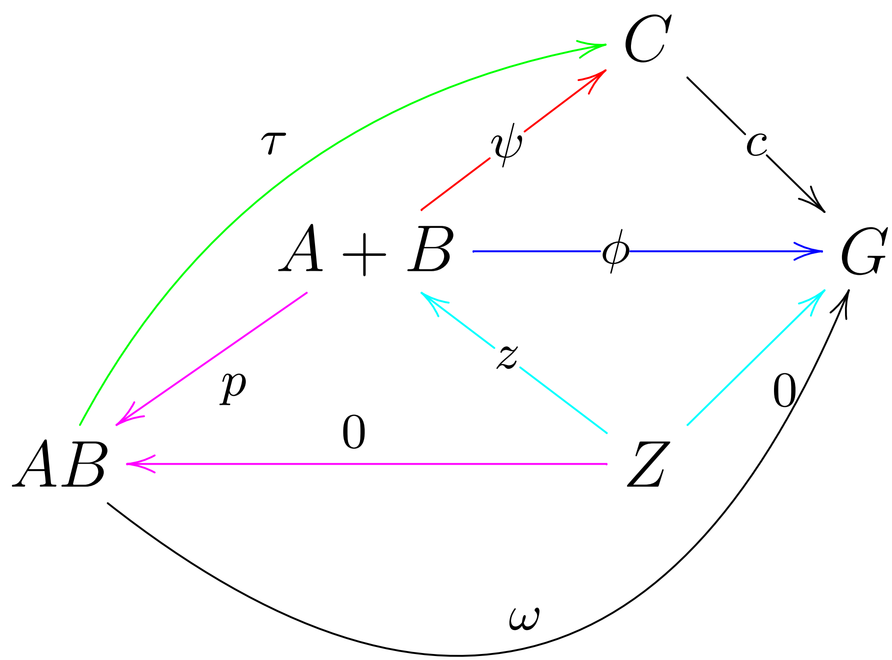
$$\phi \circ z = 0 \quad \text{(which implies that)} \tag{9}$$

$$c \circ \psi z = 0 \quad \text{(since } c \circ \psi = \phi \text{)} \tag{10}$$

$$\psi \circ z = 0 \quad \text{(since } c \text{ is monic)} \tag{11}$$

$$\tag{12}$$

Thus, there exists a unique arrow τ so that



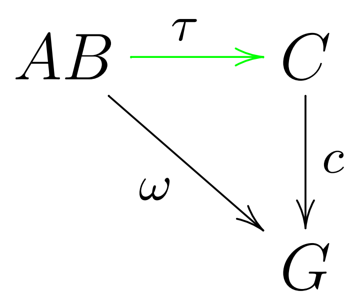
commutes - that is, so that $\tau \circ p = \psi$. (Why?)

Now,

$$c \circ \tau \circ p = \omega \circ p \qquad \text{(which implies that)} \tag{13}$$

$$c \circ \tau = \omega \qquad \text{(since } p\text{, being a cokernel, is epi)} \tag{14}$$

Thus, τ is monic **??**, and



commutes. In other words, $AB \leq C$. Since C was an arbitrary upper bound for $\{< A, a >; < B, b >\}$ we have proven that AB is the least upper bound.