

# Quiz

What is the name of the text for this course?

# Commutative Rings

Traditionally, rings are thought of as Abelian groups with some additional structure - a set with two binary operations. However, the category of commutative rings differs greatly from the category  $\text{AbG}$ . In this chapter we intend to define the definition of the category of commutative rings, which we shall denote  $\text{CR}$ , by imposing increasingly restrictive conditions until we arrive at  $\text{CR}$ . Following Luo, there will be, broadly, 9 such conditions. As we did with  $\text{AbG}$ , upon the completion of these 9 sections we shall examine what sorts of objects - understood internally- live in  $\text{CR}$ .

# Right Categories

As we saw in the last chapter,  $\text{AbG}$  has a zero object - that is, an object which is simultaneously initial and terminal. In the category of commutative rings  $\text{CR}$ , there is no such object. Like Abelian Groups, there is an object which, viewed traditionally consists of a set with one element, which we also call the zero ring. Specifically, the category of commutative rings has what is often called a “strict terminal object”

# Definition

A terminal object  $1$  in a category  $C$  is called **strict** if every arrow with  $1$  as its source is an isomorphism. The dual notion is - not surprisingly, perhaps - called a **strict initial object**

Categories with strict initial objects are called **left** categories and those with strict terminal objects **right** categories.  $\mathbf{CR}$  is a right category.

# Unitary Categories

Recall that we called a monic regular if there were a pair of arrows for which it was the equalizer. We shall also call the dual “regular” but shall be careful to differentiate between the two by always identifying an arrow as a regular *monic* or regular *epic*. To emphasize we state the following:

# Definition

An arrow is called a **regular epic** if it is a coequalizer for some pair of arrows.

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his terminology helps us identify a second defining characteristic of the category CR:

# Definition

A category is called **right unitary**

1. if it is a right category and
2. if every arrow  $t : T \rightarrow 1$  to the strict terminal object is a regular epi

# Extensive Categories

The dual category of CR shares properties with categories whose objects are best understood geometrically. SET is such a category. You have probably used Venn diagrams to understand certain facts about sets: the intersection, union and complement. For example, if the intersection - the pullback - of two sets is empty this can be expressed by showing two blobs with no overlap. Another way of saying this is to say that the pullback is initial. For those familiar with the category of topological spaces or of manifolds, the same is true. Objects understood geometrically have pullback equal to the strict initial object exactly when those objects do not overlap. We hope that this motivates the following:

# Definition

Two arrows with common source  $b : A \rightarrow B$  and  $c : A \rightarrow C$  are said to be **codisjoint** if their pushforward exists and is the strict terminal object.

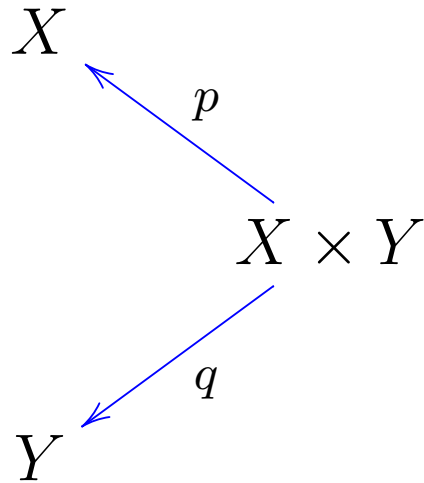
# Note:

We chose the term “codisjoint” rather than “disjoint” here since disjoint is a term with clear geometric connotations and so would be most suggestive of that in the dual category to CR. As with other concepts the dual is indicated by use of the prefix “co”.

Finite products exist in  $\mathbf{CR}$  and possess an interesting property:

# Definition

We will say that a product of objects



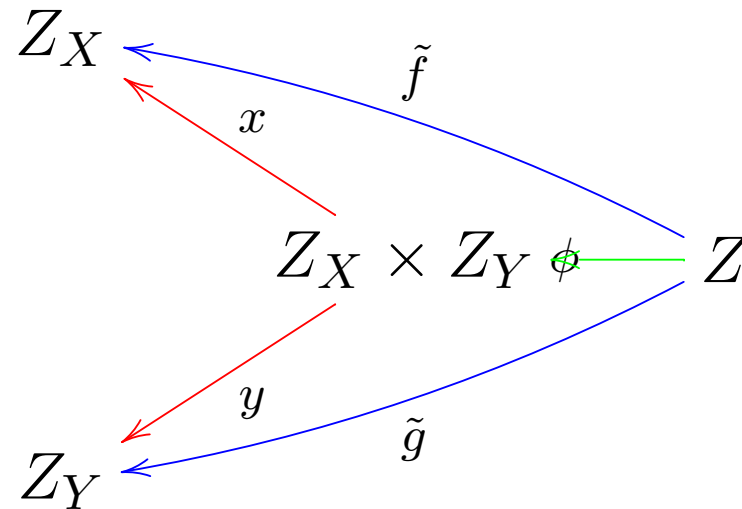
is **codisjoint** if the arrows  $p$  and  $q$  are codisjoint.

# Notation

We shall find it easier to use the following notation when making the next definition: Let  $f : X \times Y \rightarrow Z$  be an arrow. By  $Z_X$  we shall mean the pushforward of  $f$  by  $p$  the projection arrow in the product diagram above.

# Definition

We keep the notation from ???. A category is said to have **costable products** if the unique arrow  $\phi$  so that



commutes - where  $\tilde{f}$  is the pushforward of  $f$  by  $p$  and  $\tilde{g}$  the pushforward of  $g$  by  $q$ , and where the red diagram is the product diagram - is an isomorphism.

The dual notion in this case is referred to as **stable sums**.  
Most geometric categories have them:

# Definition

A right category is said to be **right extensive** if every product is codisjoint and costable.

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After you have successfully worked the exercises in this section, or else are willing to take them on faith, you will have proven that SET is left extensive. As promised we are moving toward a definition of what CR is. So far we can comprehensibly say it is right extensive.

# Rextensive Categories

Recall that we say a limit or colimit is “finite” if the number of objects in the limit or colimit diagram is finite.

# Definition

A **rextensive category** is a right extensive category in which all finite colimits exist.

# Analytic Categories

Recall that in the chapter on Abelian Groups we proved that every arrow can be uniquely factored - up to isomorphism - into the composition of a regular mono and an epi. Although this is not possible in every category, it is in CR. We make the following more general definition:

# Definition

We will refer to a category as **right analytic** if it is rextensive and if every arrow can be written as the composition of a regular epi followed by a monic.

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As you might have guessed, the dual notion is called **left analytic**. Many familiar and important categories are left analytic- including SET.

# Right Analytic Geometries

This is the section where things get a bit more interesting. Recall that we have shown that the pullback of a monic is monic - and, thus, by duality - showed that the pushforward of an epic is epic. But what about the reverse? Is the pullback of a epic epic? The pushforward of a monic monic? The answer is not always. Is it ever? The answer is - at least in **CR** - when the pushforward is by a special type of arrow.

# Definition

An arrow is said to be **pre-flat** provided that the pushforward of any mono by it is again a mono.

The dual notion is called **pre - coflat**.

# Definition

An arrow is said to be **flat** if every pushforward of it is pre-flat.

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gain the dual is referred to as **coflat**.

# Definition

An arrow  $f$  is said to **factor through**  $g$  if there exists an arrow  $h$  so that  $f = g \circ h$  or  $f = h \circ g$ .

We need this definition now, in order to define the notion of complement. The idea of complement is somewhat akin to that of that of top elements. More precisely, we have:

# Definition

Let  $e$  be an epi. The **complement** of  $e$ , which we shall often denote  $e^c$ , is an arrow satisfying the following criteria:

1.  $e$  and  $e^c$  are codisjoint
2. Every arrow which is codisjoint with  $e$  factors through  $e^c$ .

# Definition

An strong epic is said to be **codisjunctable** if its complement is flat.

# Definition

Let  $\{f_i : A \rightarrow B_i\}_{i \in I}$  be a collection of arrows and objects.  
The colimit - both the object  $U$  and unique arrow  $u : A \rightarrow U$  - of them will be referred to as **union** of the diagram.

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he dual notion is called **intersection**.

# Definition

We will say that a category is **locally codisjunctable** if every strong epic can be written as the union of codisjunctable epis.

# Definition

A category is said to be **perfect** if every union of strong epis exist.

# exercise

Show that SET is perfect.

# exercise

Show that AbG is perfect.

# Definition

An arrow is said to be **nilpotent** if the only arrow with which it is codisjoint is one whose target is the terminal object.

# Exercise

Show that the identity arrow for any non terminal object is nilpotent.

# Definition

The dual notion to nilpotent is called **unipotent**. That is, an arrow is **unipotent** if the only arrow with which it is disjoint is one whose source is the initial object.

# Exercise

Show that in **SET** a function is unipotent if and only if it is onto.

# Definition

An object is said to be **reduced** if any nilpotent arrow to it is monic.

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he fact that such objects will be of interest to us is, in part, what distinguishes CR from SET. Indeed, we have:

# exercise

Show that there every object in `SET` is reduced.

# Definition

A category is said to be **reduced** if every non-terminal object has a reduced quotient object.

Obviously SET is a perfect. Based on what we know about AbG, is it too perfect?

# Definition

A right analytic category which is locally codisjunctable reducible and perfect is called a **right analytic geometry**