Quiz

Define $strongmono$.

Last Time

- Pullbacks exist in $\mathbf{SET}$

Last Time

- Pullbacks exist in $\mathbf{SET}$
- Pullback of a pullback

Definition:

An arrow $f : A \longrightarrow B$ is said to be a $strongmono$ if no pullback of it is a non-isomorphic epi.

Theorem

An arrow $f : A \longrightarrow B$ is an isomorphism if and only if it is both a strong mono and an epi.

Proof

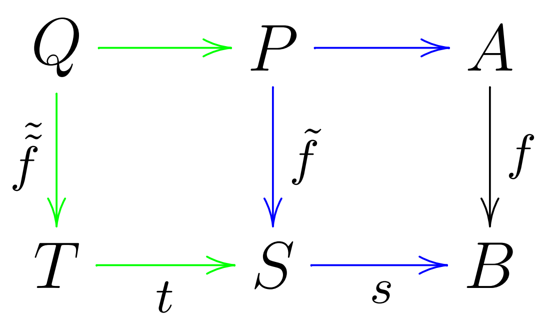
Suppose f is an isomorphism. Then, we have seen that its pullback is an isomorphism. In particular it can never be a non-isomorphic epi. It is also a fact that every isomorphism is an epi. Thus, it is both a strong mono and an epi. On the other hand if f is both a strong mono and an epi, then its pullback by 1_B is again f, which means it must be an isomorphism (since no pullback can be a non-isomorphic epi, and f is epi)

Lemma

The pullback of a strong mono is also a strong mono.

Proof

Let $f : A \to B$ be a strong mono and $\tilde{f}$ its pullback by some arrow s. Let $\tilde{\tilde{f}}$ be the pullback of $\tilde{f}$ by some arrow t. Then we have seen that $\tilde{\tilde{f}}$ is the pullback of f by $s \circ t$. Thus, $\tilde{\tilde{f}}$ cannot be a non-isomorphic epi, since f is a strong mono. Thus, $\tilde{f}$ is a strong mono.



Not all categories have pullbacks or pushforwards. The following definition for strong mono can be substituted for the above in case we are in such a category. This equivalence will also make some future proofs more transparent.

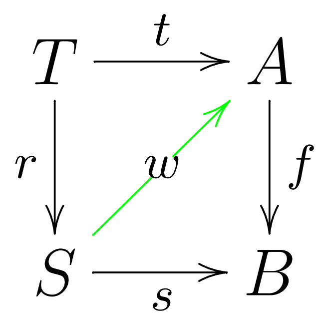
Lemma

A mono $f : A \to B$ is strong if and only if for every commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

in which r is an epic arrow, there exists a unique map w

so that in the diagram



$f \circ w = s$ and $w \circ r = t$.

Proof

Suppose $f : A \to B$ is strong. Let s be an arbitrary arrow and r an epic arrow so that

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

commutes.

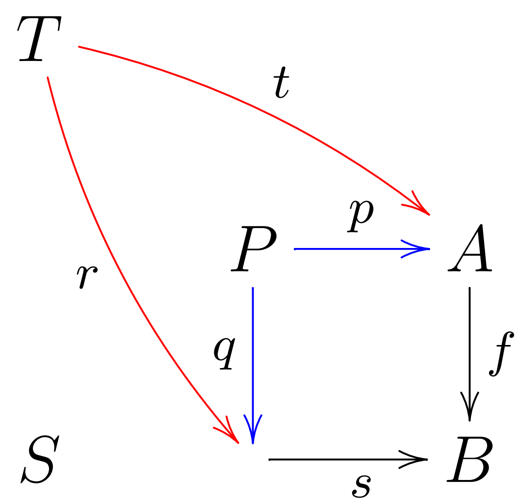
Let

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

be the pullback of f by s.

Then,

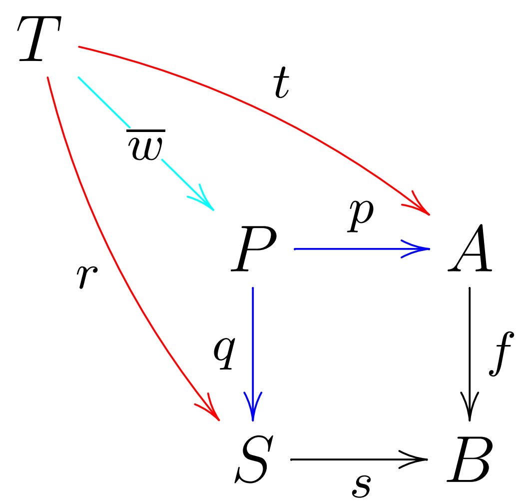
the diagram



commutes.

Thus,

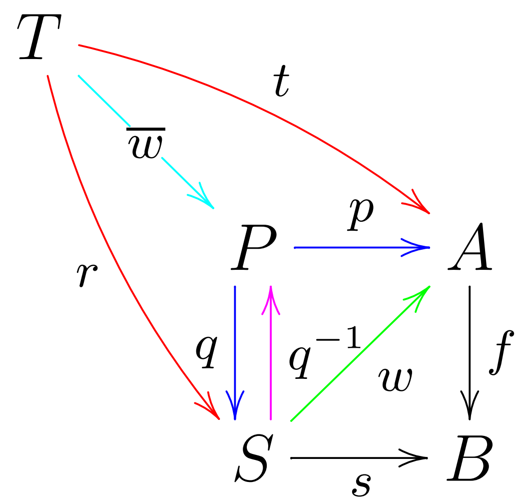
there exists a uniqe $\overline{w}$ so that



commutes. We have already proven that since r is epi, q must be as well.

Denote

by q^{-1} the inverse arrow of q. Set $w = p \circ q^{-1}$.



Now, $f \circ w = f \circ p \circ q^{-1} = s \circ q \circ q^{-1} = s$

$w \circ r = p \circ q^{-1} \circ r = p \circ \overline{w} = t$.

(Since $q \circ \overline{w} = r \Rightarrow q^{-1} \circ q \circ \overline{w} = q^{-1} \circ r \Rightarrow \overline{w} = q^{-1} \circ r$)

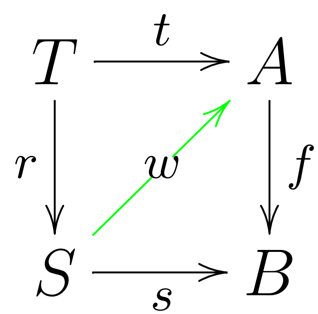
Conversely

Suppose that f is a monic arrow so that for every commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

in which r is an epic arrow, there exists a unique map w

so that in the diagram



$f \circ w = s$ and $w \circ r = t$

Let k be any arrow

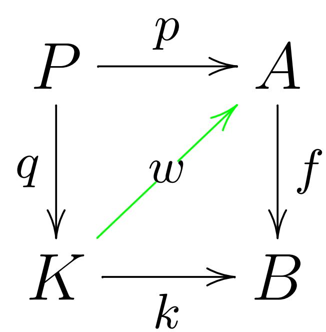
and

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ K & \xrightarrow[k]{} & B \end{array}$$

be the pullback of f by k.

Suppose that

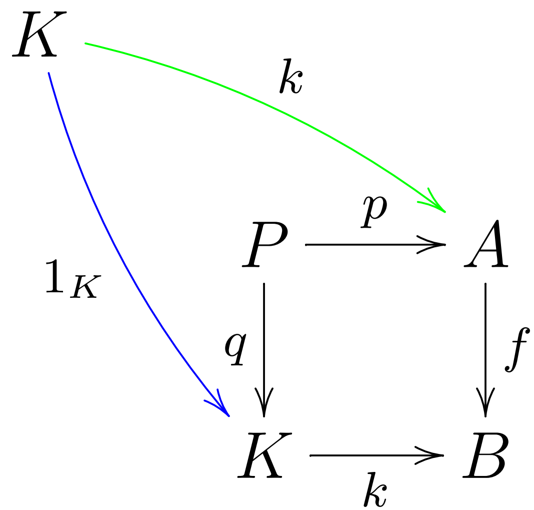
k is epic. Then, by hypothesis, there exists a unique w so that



$f \circ w = k$ and $w \circ q = p$.

Then,

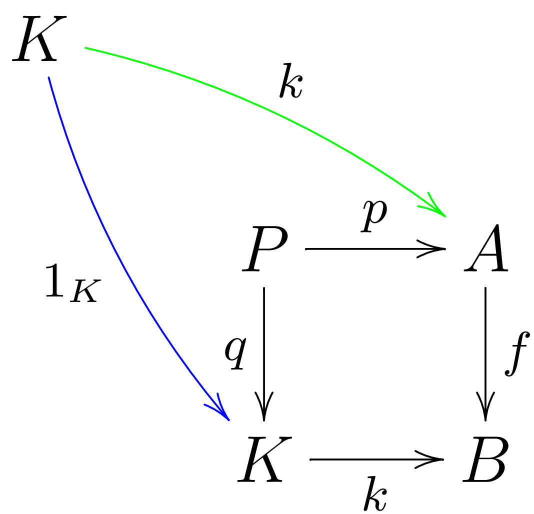
the diagram



commutes.

Thus,

there exists a unique arrow ϕ so that



commutes. Since f is monic, so too is q. Since $q \circ \phi = 1_K$, q is an isomorphism.

Next Time

Read pages 20-25