Quiz

Define $pullback$.

Last Time

We proved:

- The pullback of an isomorphism is an isomorphism

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- The pullback of an isomorphism is an isomorphism
- The pullback of a mono is a mono

This Time:

- The existence of pullbacks in $\mathbf{SET}$

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- The pullback of a pullback

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- The existence of pullbacks in $\mathbf{SET}$
- The pullback of a pullback
- Strong Monos

Lemma

Pullbacks always exist in $\mathbf{SET}$.

proof

Let

$$\begin{array}{ccc} & & X \\ & & \downarrow f \\ Y & \xrightarrow[g]{} & Z \end{array}$$

be a diagram in $\mathbf{SET}$.

Define

the set
$P := \{(x, y) \in X \times Y | f(x) = g(y)\}$.

with functions

$p : P \longrightarrow X$ defined by $(x, y) \mapsto x$
$q : P \longrightarrow Y$ defined by $(x, y) \mapsto y$.

Then,

the diagram

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

commutes:

Let

$$(x, y) \in P$$

. Then,

$$f \circ p(x, y) = f(x)$$

and

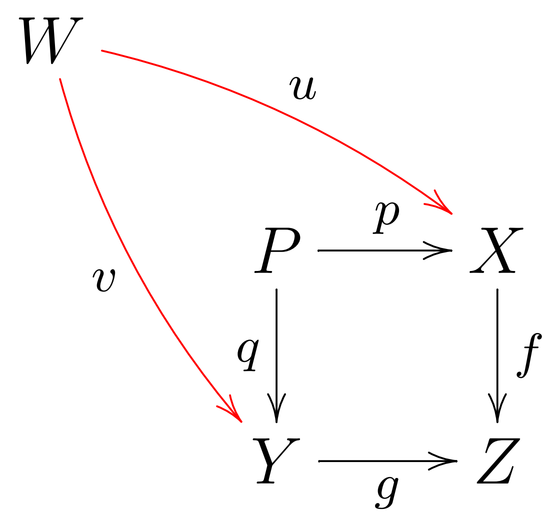
$$g \circ q(x, y) = g(y)$$

But, remember,

$(x, y) \in P$ iff $f(x) = g(y)$. Thus, the diagram commutes.

Now,

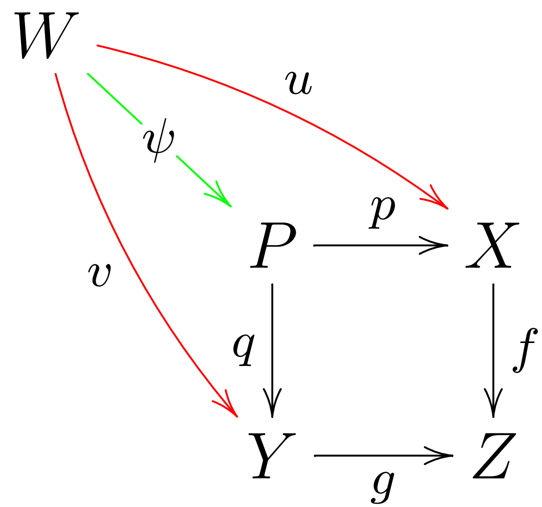
suppose we have a set W and functions u and v so that



commutes.

Set

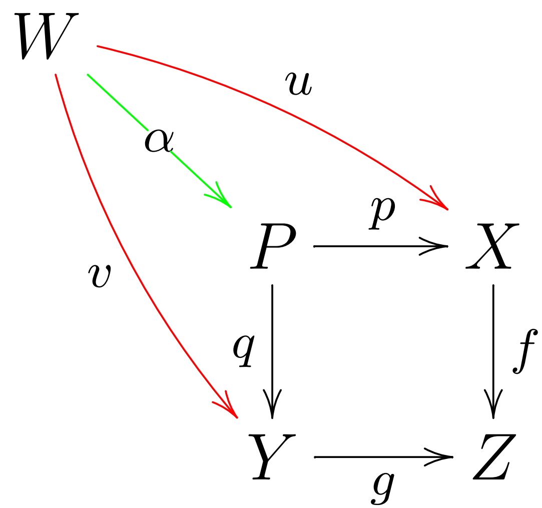
$\psi : W \to P$ by $w \mapsto (u(w), v(w))$. Then,



commutes.

Uniqueness:

Suppose



commutes.

Where

$\alpha(w) =< x, y >$. Now, $p(\alpha(w)) = u(w)$ and $q(\alpha(w)) = v(w)$. This means, however, that $\alpha(w) =< u(w), v(w) >= \psi(w)$. Thus, ψ is unique.

Lemma:

Let $f : A \to B$ and $s : S \to B$ be arrows. Let

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

be the pullback diagram. Let $t : T \to S$ be any arrow and

$$\begin{array}{ccc} Q & \xrightarrow{v} & P \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow[t]{} & S \end{array}$$

be the pullback of p by t.

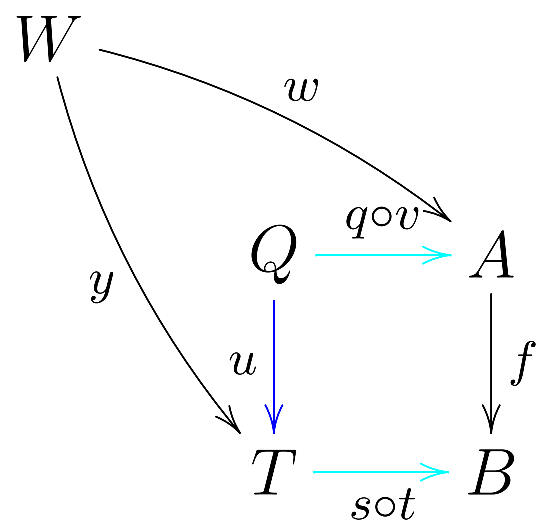
Then,

$$\begin{array}{ccc} Q & \xrightarrow{q \circ v} & A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[s \circ t]{} & B \end{array}$$

is also a pullback diagram.

Proof

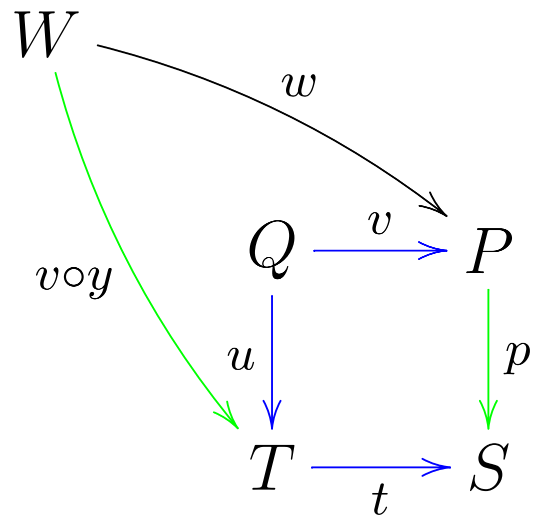
Let W be any object and w and y any arrows with source W such that



commutes.

Then,

so does



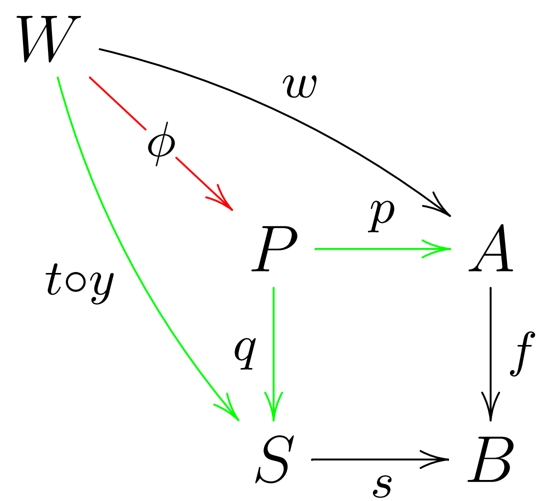
Since

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

is a pullback diagram,

there exists

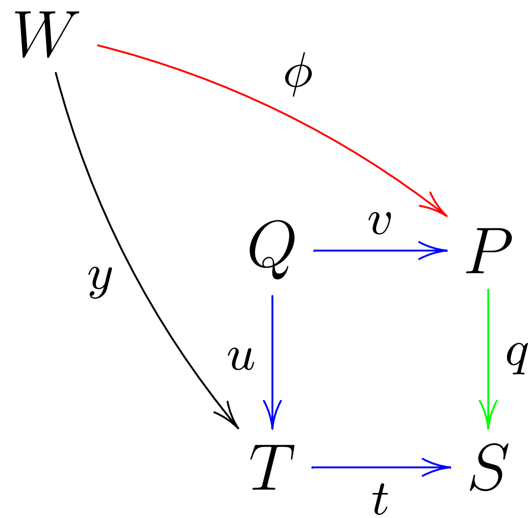
a unique arrow ϕ so that



commutes.

Thus,

we have a commutative diagram



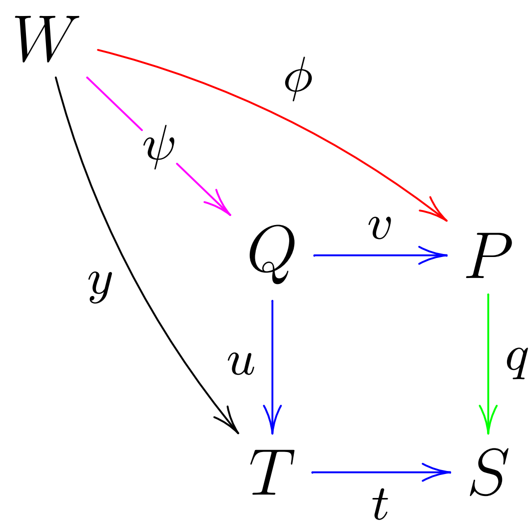
Since

$$\begin{array}{ccc} Q & \xrightarrow{v} & P \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle q} \\ T & \xrightarrow[t]{} & S \end{array}$$

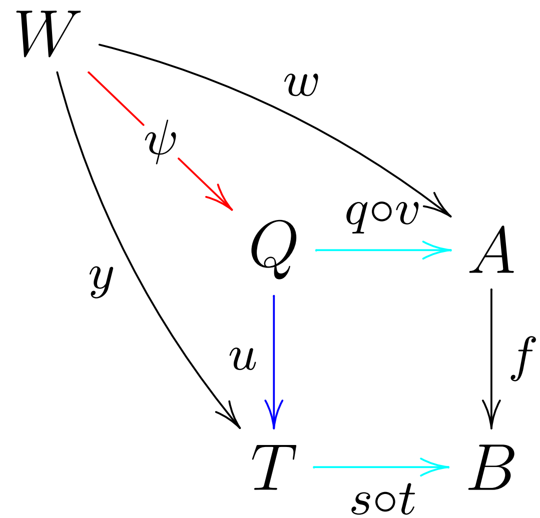
is a pullback diagram,

There

exists a uniqe arrow ψ so that



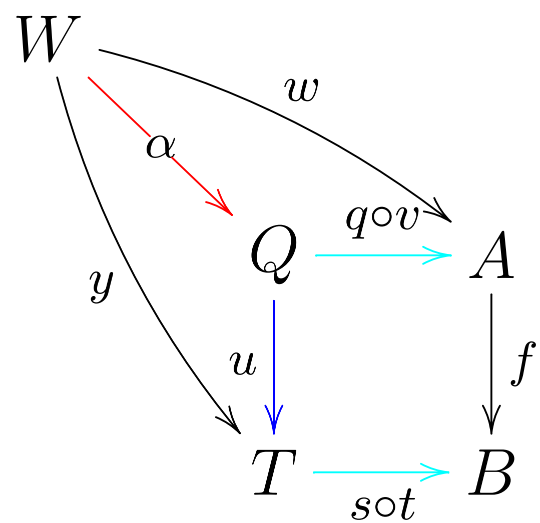
commutes.



commutes.

Suppose

now that



also commutes.

Then,

by the uniqueness of ϕ,

$$v\alpha = \phi$$

But, since $u \circ \alpha = y$ by assumption, we must have that $\alpha = \psi$ by the uniqueness of ψ.

Definition:

An arrow $f : A \longrightarrow B$ is said to be a $strongmono$ if no pullback of it is a non-isomorphic epi.

Theorem

An arrow $f : A \longrightarrow B$ is an isomorphism if and only if it is both a strong mono and an epi.

Proof

Suppose f is an isomorphism. Then, we have seen that its pullback is an isomorphism. Thus, it is both a strong mono and an epi. On the other hand if f is both a strong mono and an epi, then its pullback by 1_B is again f, which means it must be an isomorphism (since no pullback can be a non-isomorphic epi, and it is epi)

Lemma

The pullback of a strong mono is also a strong mono.

Proof

Let $f : A \to B$ be a strong mono and $\tilde{f}$ its pullback by some arrow s. Let $\tilde{\tilde{f}}$ be the pullback of $\tilde{f}$ by some arrow t. Then we have seen that $\tilde{\tilde{f}}$ is the pullback of f by $s \circ t$. Thus, $\tilde{\tilde{f}}$ cannot be a non-isomorphic epi, since f is a strong mono. Thus, $\tilde{f}$ is a strong mono.

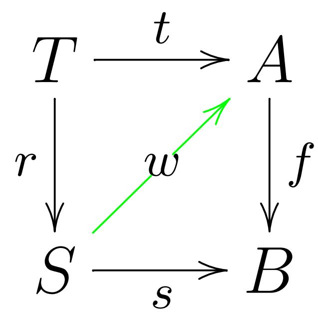
Not all categories have pullbacks or pushforwards. The following definition for strong mono can be substituted for the above in case we are in such a category:

Lemma

A mono $f : A \to B$ is strong if and only if for every commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

with r a monic arrow, there exists a unique map w



$f \circ w = s$ and $w \circ r = t$.

Proof

Suppose $f : A \to B$ is strong. Let s be an arbitrary arrow and r a monic arrow so that

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

commutes.

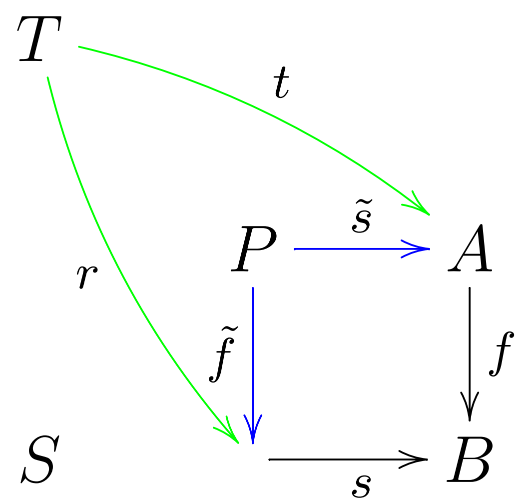
Let

$$\begin{array}{ccc} P & \xrightarrow{\tilde{s}} & A \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

be the pullback of f by s.

Then,

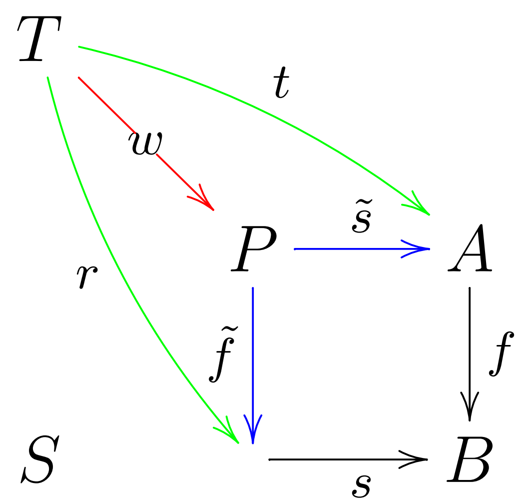
the diagram



commutes.

Thus,

there exists a uniqe w so that



Note

that this implies that