Quiz

Define *pullback*
Last Time

Suppose that

is a pullback diagram. If $f$ is an isomorphism, then so is $\tilde{f}$. 
Proof

Since $f : A \rightarrow B$ is an isomorphism, there exists $g : B \rightarrow A$ so that $f \circ g = 1_B$ and $g \circ f = 1_A$. Consider the commutative diagram
By definition we have a unique map $\phi$ so that

$$S \xrightarrow{\phi} P \twoheadrightarrow A$$

Thus, $\tilde{f} \circ \phi = 1_S$. 

$$S \xrightarrow{1_S} P \xrightarrow{\tilde{s}} A$$

$$S \xrightarrow{\tilde{f}} B$$
Now, consider the diagram:

\[ \begin{array}{c}
P \\
\downarrow \tilde{f} \\
\tilde{g} \downarrow \\
S \quad \tilde{s} \quad f \\
\end{array} \quad \begin{array}{c}
P \overset{\tilde{s}}{\longrightarrow} A \\
\tilde{s} \downarrow \\
S \overset{s}{\longrightarrow} B \\
\end{array} \]
We see then that $P \sim f \sim P \sim f \sim A$ commutes.
But,

so too does
The definition of pullback says, however, that there can be only one map from $P$ to $P$ making that diagram commute. Thus, $\phi \circ \tilde{f} = 1_P$. Since we have now that both $\tilde{f} \circ \phi = 1_S$ and $\phi \circ \tilde{f} = 1_P$, we have that $\tilde{f}$, the pullback of $f$ by an arbitrary map is an isomorphism. Which is what we hoped to show.
Lemma:

The pullback of a mono by any map is again a mono.
proof

Suppose $f : A \to B$ is a mono and $g : C \to B$ is any map. Let

\[
\begin{array}{ccc}
P & \xrightarrow{\tilde{g}} & A \\
\tilde{f} & & f \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & B
\end{array}
\]

be the pullback diagram.
Suppose that \( \tilde{f} \circ s = \tilde{f} \circ t \). Then, \( g \circ \tilde{f} s = g \circ \tilde{f} t \) which by the commutativity of the pullback diagram implies that \( f \circ \tilde{g} s = f \circ \tilde{g} t \). Since \( f \) is monic, we thus have that \( \tilde{g} \circ s = \tilde{g} \circ t \).
In other words

In other words, the diagram

\[ S \xrightarrow{\tilde{g} \circ s = \tilde{g} \circ t} P \xrightarrow{\tilde{g}} A \]
\[ C \xrightarrow{g} B \]

commutes.
Since

Since \( P \xrightarrow{\tilde{g}} A \)
\( \tilde{f} \)
\( C \xrightarrow{g} B \)

is a pullback,
there exists

a unique $\phi$ so that

commutes.
Since both

\[ S \xrightarrow{\tilde{g} \circ s = \tilde{g} \circ t} \tilde{g} \circ A \]

\[ P \xrightarrow{\tilde{f} \circ s = \tilde{f} \circ t} C \xrightarrow{\tilde{f}} A \]

\[ C \xrightarrow{g} B \]
and

\[ S \xrightarrow{\tilde{g} \circ s = \tilde{g} \circ t} \]

\[ \tilde{f} \circ s = \tilde{f} \circ t \]

\[ P \xrightarrow{\tilde{g}} A \]

\[ \tilde{f} \]

\[ C \xrightarrow{g} B \]

commute
we must have that \( s = t = \phi \). Thus, since \( s \) and \( t \) were arbitrary, \( \tilde{f} \) is monic.
This leaves the question:

Is the pullback of an epi also an epi?
The Answer is

NO.
To See This

We will have to provide an example of a pullback.