# Quiz

Define $pullback$

# Last Time

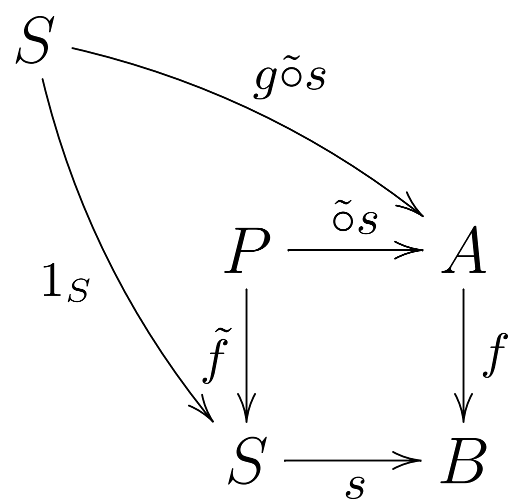
Suppose that

$$\begin{array}{ccc} P & \xrightarrow{\tilde{s}} & A \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[s]{} & C \end{array}$$

is a pullback diagram. If $f$ is an isomorphism, then so is $\tilde{f}$.

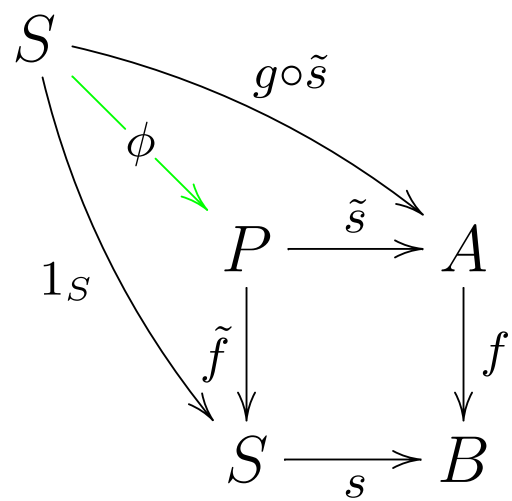
# Proof

Since $f : A \longrightarrow B$ is an isomorphism, there exists $g : B \longrightarrow A$ so that $f \circ g = 1_B$ and $g \circ f = 1_A$. Consider the commutative diagram



# By definition

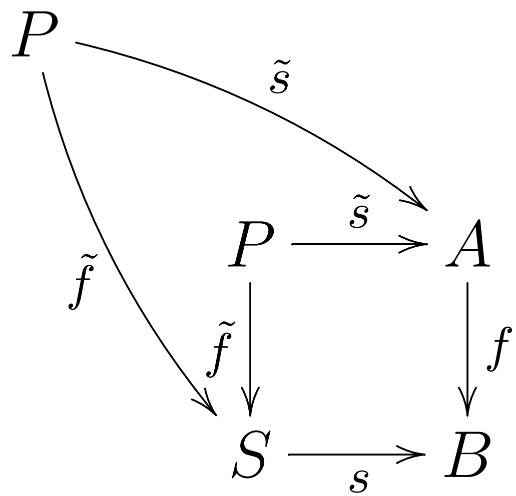
we have a uniqe map $\phi$ so that




Thus, $\tilde{f} \circ \phi = 1_S$.

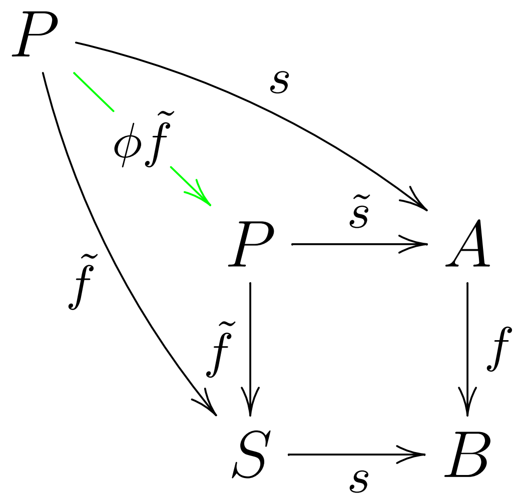
# Now,

Now consider the diagram



# We

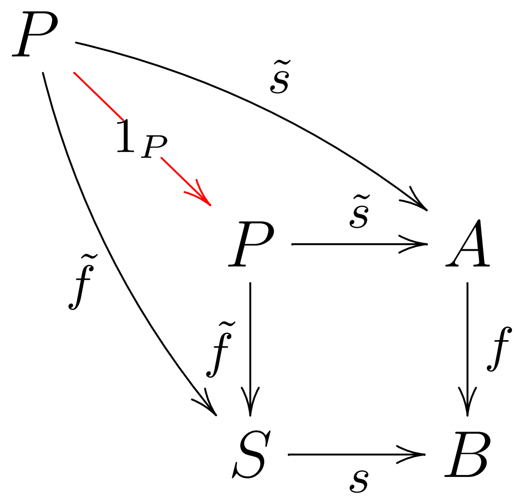
see then that




commutes.

# But,

so too does



The definition of pullback says, however, that there can be only one map from $P$ to $P$ making that diagram commute. Thus, $\phi \circ \tilde{f} = 1_P$. Since we have now that both $\tilde{f} \circ \phi = 1_S$ and $\phi \circ \tilde{f} = 1_P$, we have that $\tilde{f}$, the pullback of $f$ by an arbitrary map is an isomorphism. Which is what we hoped to show.

# Lemma:

The pullback of a mono by any map is again a mono.

# proof

Suppose $f : A \to B$ is a mono and $g : C \to B$ is any map. Let

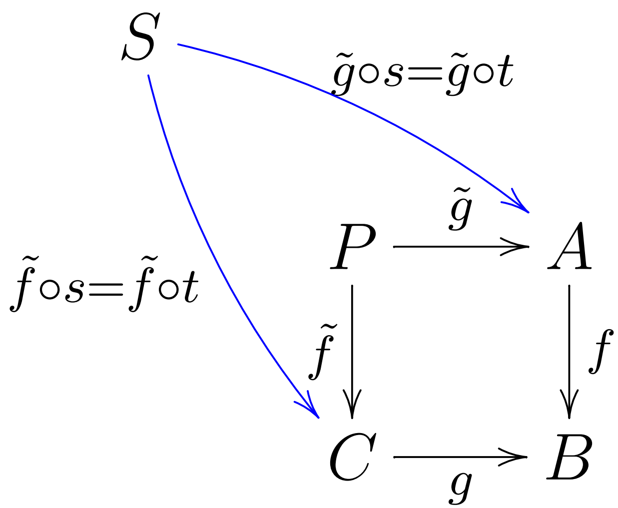
$$\begin{array}{ccc} P & \xrightarrow{\tilde{g}} & A \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & B \end{array}$$

be the pullback diagram.

Suppose that $\tilde{f} \circ s = \tilde{f} \circ t$. Then, $g \circ \tilde{f}s = g \circ \tilde{f}t$ which by the commutativity of the pullback diagram implies that $f \circ \tilde{g}s = f \circ \tilde{g}t$. Since $f$ is monic, we thus have that $\tilde{g} \circ s = \tilde{g} \circ t$.

# In other words

In other words, the diagram




commutes.

# Since

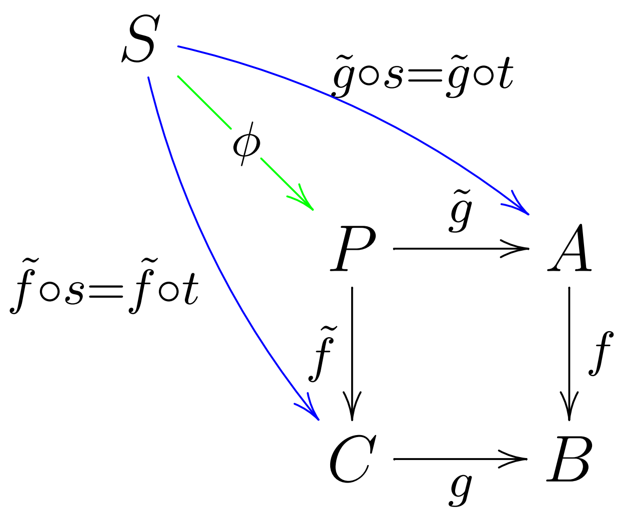
Since

$$\begin{array}{ccc} P & \xrightarrow{\tilde{g}} & A \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & B \end{array}$$

is a pullback,

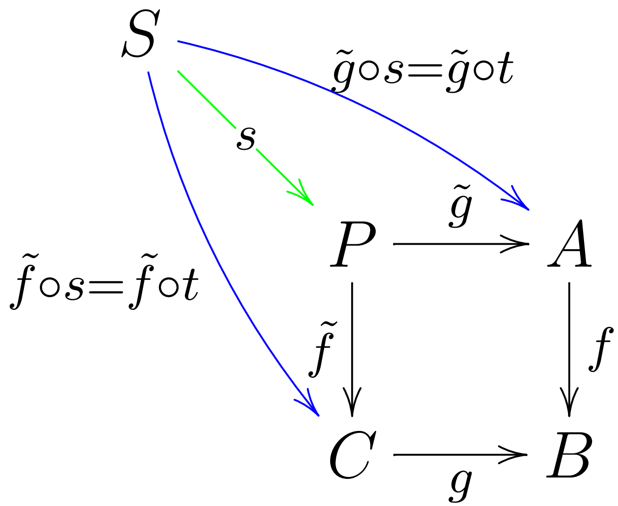
# there exists

a unique $\phi$ so that



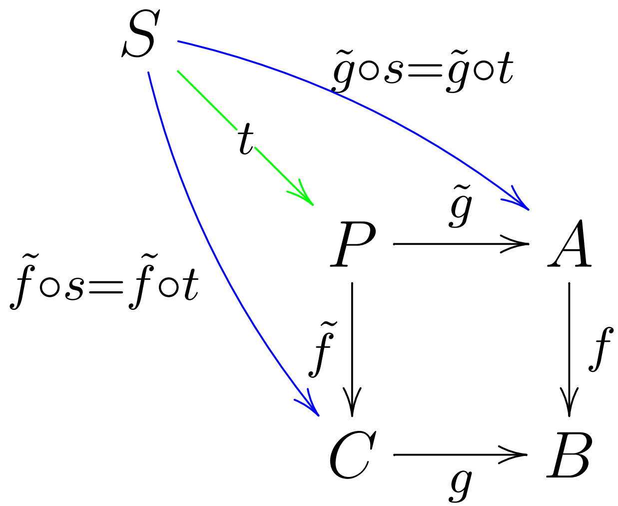

commutes.

# Since both



$S$
$\tilde{g}\circ s = \tilde{g}\circ t$
$s$
$\tilde{g}$
$P$
$A$
$\tilde{f}\circ s = \tilde{f}\circ t$
$\tilde{f}$
$f$
$C$
$B$
$g$

# and




commute

we must have that $s = t = \phi$. Thus, since $s$ and $t$ were arbitrary, $\tilde{f}$ is monic.

# This leaves the question:

Is the pullback of an epi also an epi?

# The Answer is

NO.

# To See This

We will have to provide an example of a pullback.