Quiz

What does it mean to say that
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[q]{} & D \end{array}$$
$commutes$?

Last Time:

We defined

- category

Last Time:

We defined

- category
- monic

Last Time:

We defined

- category
- monic
- epi

Last Time:

We defined

- category
- monic
- epi
- isomorphism

This Time:

- Some facts about monics and epis

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- Initial and Terminal Objects

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- Some facts about monics and epis
- Initial and Terminal Objects
- Pullbacks and Pushforwards

Fact

Suppose $h = g \circ f$ is a monic arrow. Then f is also a monic arrow.

Proof

Suppose m and n are arrows such that $f \circ m = f \circ n$. Then, $g \circ f \circ m = g \circ f \circ n$.

Thus, $h \circ m = h \circ n$. But, remember, h is monic.

Thus,

$m = n$. Since we assumed only that m and $n$were arrows such that $f \circ m = f \circ n$, f must be monic.

Dual Statement:?

‘Suppose $h = g \circ f$ is epi. Then g is epi.

Universals

Generally, and imprecisely, a universal in a category is some object and, possibly, collection of maps which possesses some property which is uniqe up to isomorphism

When we say that an object is “unique up to isomorphism” we mean that any two objects with the same property must be isomorphic.

Examples

and their duals:

- initial - terminal objects

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and their duals:

- initial - terminal objects
- pullbacks - pushforwards
- equalizers - coequalizers
- products - coproducts

An object I in a category $\mathbf{C}$ is called $initial$ if for every object A in $\mathbf{C}$ there is exactly one arrow $i_A : I \longrightarrow A$.

The Dual Notion:

is called a $terminal$ object.

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- as a category it will have an $initial$ object, if as a partially ordered set it has

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Examples:

If we regard a partially ordered set as a category,

- as a category it will have an $initial$ object, if as a partially ordered set it has
- a $bottom$ element
- as a category it will have a $terminal$ object, if as a partially ordered set it has
- a top element.

It is an Exercise

to show that in $\mathbf{SET}$, the empty set is initial and the set with one element is terminal.

Pullbacks

Let

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle a} \\ B & \xrightarrow[b]{} & C \end{array}$$

be a diagram in a category $\mathbf{C}$.

A Pullback

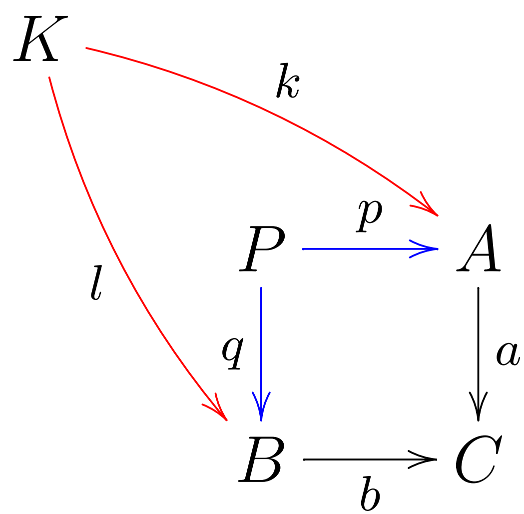
is an object P and arrows $p : P \to A$ and $q : P \to B$ so that

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow[b]{} & C \end{array}$$

with the following universal property:

Whenever

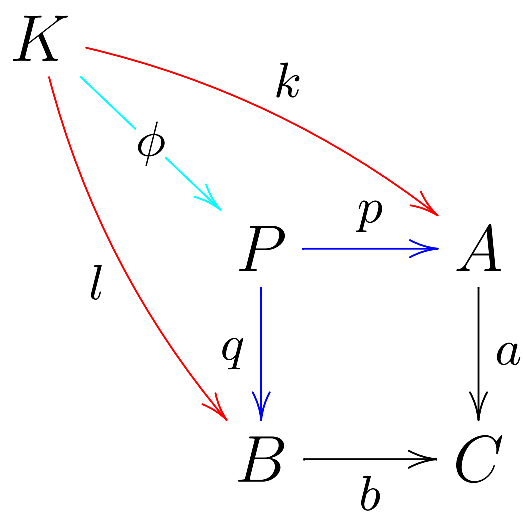
there exists an object and arrows so that



commutes.

There Exists

a unique map ϕ so that



commutes.

The dual notion

is usually called a $pushforward$.

Lemma

Suppose that

$$\begin{CD} P @>p>> A \\ @VqVV @VVaV \\ B @>>b> C \end{CD}$$

is a pullback diagram. If a is an isomorphism, then so is q.