What does it mean to say that $A \xrightarrow{f} B$ \quad \xrightarrow{g} \quad C \xrightarrow{h} \quad commutes$?
\[ h \circ f = g \]
In general,

and less precisely,
A diagram *commutes* when following any two paths between two objects in a diagram gives equal arrows.
Category Theory

We start by defining a category:
Category is a collection $A, B, C, ...$ of objects and for each pair of objects $A, B$ a set $\text{Hom}(A, B)$ of arrows.
so that:

For any objects $A, B, C$ there exists an assignment $\text{Hom}(A, B) \times \text{Home}(B, C) \rightarrow \text{Hom}(A, C)$, $f \times g \mapsto g \circ f$. If such exists $f$ and $g$ are said to be composable.
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- $h \circ (g \circ f) = (h \circ g) \circ f$ for all composable arrows $f, g, h$.

- For every object $A$ there exists an arrow $1_A$ so that if $f : A \rightarrow B$, $f \circ 1_A = f$ and if $g : C \rightarrow A$, $1_A \circ g = g$. 
Examples:

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- $C^1$ whose only object is $\mathbb{R}$ and whose arrows are continuous functions $f : \mathbb{R} \to \mathbb{R}$.
Notation

An arrow $f$ between objects $A$ and $B$ will be depicted, literally, as a labeled arrow:

$$A \xrightarrow{f} B$$
Extending the metaphor, given an arrow $A \xrightarrow{f} B$ we will often refer to $B$ as the target of $f$ and to $A$ as the source.
Depicting arrows in this way proves to be one of the many nice things about category theory
For example:

The definition of the identity arrow $1_A$ can be depicted by noting that the diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\downarrow^f & & \downarrow^f \\
B & & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\uparrow^g & & \uparrow^g \\
C & & \\
\end{array}
\]

*commute* for all $f$ and $g$. 
As we shall see, properties of the arrows themselves will help to characterize the relationship between the objects which are the source and target.
For example:

An arrow $f$ is said to be \textit{monic} if whenever $f \circ g = f \circ h$, it must be that $g = h$. 
Here, for the first time, we meet duality in a categorical context:
An arrow $f$ is said to be *epi* if whenever $g \circ f = h \circ f$, it must be that $g = h$. 
In what sense are monic and epic arrows “dual”? 
To say

\[ f \circ g = f \circ h \]

Is to say that the diagram

\[ \begin{array}{ccc}
C & \xrightarrow{g} & A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow & & \\
\end{array} \]

commutes.
To say

\[ g \circ f = h \circ f \]

Is to say that the diagram

\[
\begin{array}{c}
C \overset{h}{\leftarrow} B \overset{f}{\leftarrow} A \\
\end{array}
\]

commutes.
The diagrams

\[ C \xrightarrow{g} A \xrightarrow{f} B \]

and

\[ C \xleftarrow{h} B \xleftarrow{f} A \]

are the same, but with the arrows reversed.
Examples

Any 1–1 function is monic. Any onto function is epic.
A third type of arrow is the *isomorphism*. 
Definition

An arrow $f : A \to B$ is an isomorphism if there exists an arrow $g : B \to A$ so that

$f \circ g = 1_B$
$g \circ f = 1_A$. 
Example:

A function of sets is an isomorphism if and only if it is bijective.
The elegance of category theory comes at a price: In category theory, isomorphisms are generally as close as we can get to equality.
For example

In $\text{SET}$, the sets consisting of the three blind mice and the three stooges are isomorphic. So, from the perspective of the category theorist, they are the same.