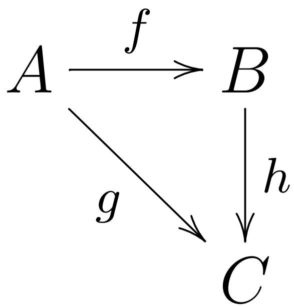
Quiz

What does it mean to say that



commutes?

Answer

$h \circ f = g$

In general,

and less precisely,
A diagram $commutes$ when following any two paths between two objects in a diagram gives equal arrows.

Category Theory

We start by defining a category:

A

$Category$ is a collection $A, B, C, ...$ of objects and for each pair of objects A, B a set $Hom(A, B)$ of $arrows$

so that:

- For any objects A, B, C there exists an assignment $Hom(A, B) \times Home(B, C) \longrightarrow Hom(A, C)$, $f \times g \mapsto g \circ f$. If such exists f and g are said to be $composable$.

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- $h \circ (g \circ f) = (h \circ g) \circ f$ for all composable arrows f, g, h.
- For every object A there exists an arrow 1_A so that if $f : A \rightarrow B$, $f \circ 1_A = f$ and if $g : C \rightarrow A$, $1_A \circ g = g$.

Examples:

- $\mathbf{SET}$ whose objects are sets and whose arrows are functions

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- $\mathbf{C}^1$ whose only object is $\mathbb{R}$ and whose arrows are continuous functions $f : \mathbb{R} \to \mathbb{R}$.

Notation

An arrow f between objects A and B will be depicted, literally, as a labeled arrow:

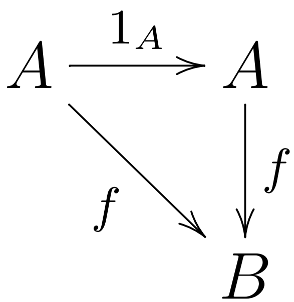
$$A \xrightarrow{f} B$$

Extending the metaphor, given an arrow $A \xrightarrow{f} B$ we will often refer to B as the $target$ of f and to A as the $source$.

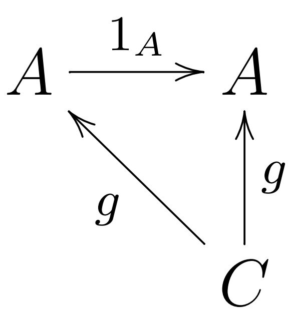
Depicting arrows in this way proves to be one of the many nice things about category theory

For example:

The definition of the identity arrow 1_A can depicted by noting that the diagrams



and



commute for all f and g.

As we shall see, properties of the arrows themselves will help to characterize the relationship between the objects which are the source and target.

For example:

An arrow f is said to be $monic$ if whenever $f \circ g = f \circ h$, it must be that $g = h$.

Here, for the first time, we meet duality in a categorical context:

An arrow f is said to be epi if whenever $g \circ f = h \circ f$, it must be that $g = h$.

Question

In what sense are monic and epic arrows “dual”?

To say

$f \circ g = f \circ h$

Is to say that the diagram

$$C \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

commutes.

To say

$g \circ f = h \circ f$

Is to say that the diagram

$$C \underset{g}{\overset{h}{\leftleftarrows}} B \xleftarrow{f} A$$

commutes.

The diagrams

$$C \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

and

$$C \underset{g}{\overset{h}{\leftleftarrows}} B \xleftarrow{f} A$$

are the same, but with the arrows reversed.

Examples

Any $1-1$ function is monic. Any onto function is epic.

A third type of arrow is the $isomorphism$.

Definition

An arrow $f : A \longrightarrow B$ is an $isomorphism$ if there exists an arrow $g : B \longrightarrow A$ so that
$f \circ g = 1_B$
$g \circ f = 1_A$.

Example:

A function of sets is an isomorphism if and only if it is bijective.

The elegance of category theory comes at a price: In category theory, isomorphisms are generally as close as we can get to equality.

For example

In $\mathbf{SET}$, the sets consisting of the three blind mice and the three stooges are isomorphic. So, from the perspective of the category theorist, they are the same.