Quiz

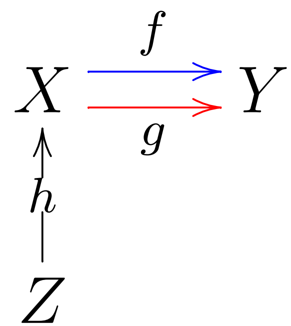
Let $f, g : A \to B$ be two arrows. Define the $equalizer$ of f and g.

Lemma

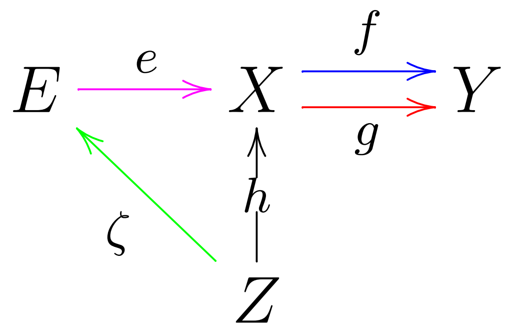
Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be two functions. Then the equalizer of f and g is the set $E := \{x \in X | f(x) = g(x)\}$ and the function $e : E \longrightarrow X$ defined by $e(w) = w$ - that is , the inclusion.

Proof

First, $Check\ that\ fe = ge$. Now, let $h : Z \to X$ be a function such that



commutes. In this case, this is just a different way of saying that $f(h(z)) = g(h(z))$ for every $z \in Z$. This is important, though, because it insures



$commuteswhere\ \zeta$ is defined by the assignment $z \mapsto h(z)$.
The proof is completed by working the following exercises and the more general lemma which follows them.

- Prove the first statement in red in the proof above

- Prove the first statement in red in the proof above
- Prove the second statement in red in the proof above. Include in your argument that the definition of ζ makes sense. (What is the issue here?)

Lemma

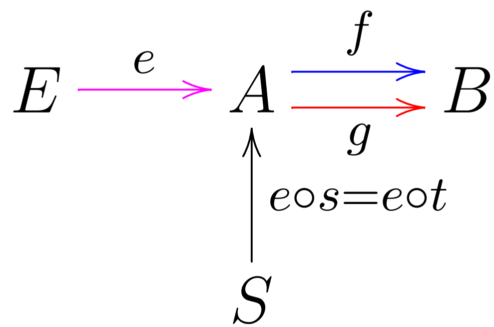
Let

$$E \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be an equalizer diagram. Then e is monic.

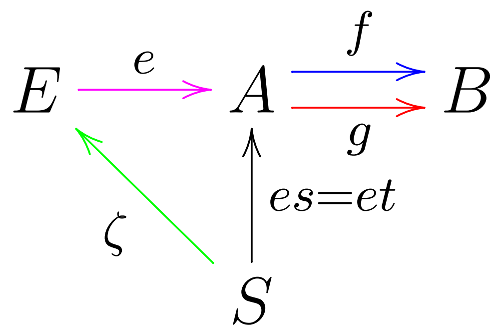
Proof

Suppose $e \circ s = e \circ t$ for a pair of arrows s and t from S to E. Then, of course, $f \circ e \circ s = g \circ e \circ s = g \circ e \circ t = f \circ e \circ t$. In other words



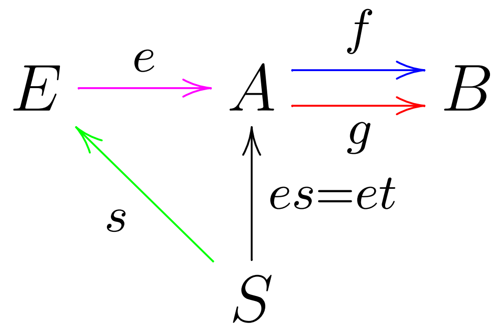
commutes

Since E is an equalizer, there exists a unique map ζ such that

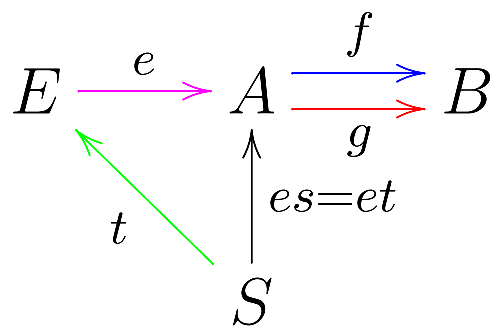


commutes.

But, both



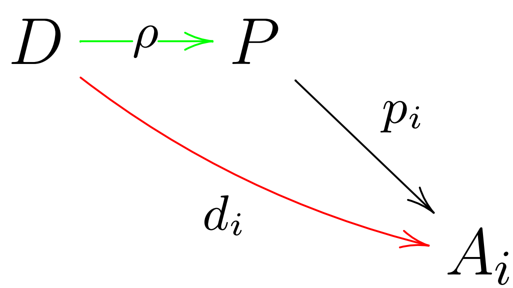
and



commute. Thus $\zeta = s = t$. Thus, e is monic.

Definition

Let $\{A_i\}_{i \in I}$ be a collection of objects. The *product* of these objects, if it exists, is defined to be an object P and arrows $\{p_i : P \to A_i\}_{i \in I}$ with the following property: For any object D and arrows $\{d_i : D \to A_i\}$ there is a unique map ρ so that for every $i \in I$



commutes.

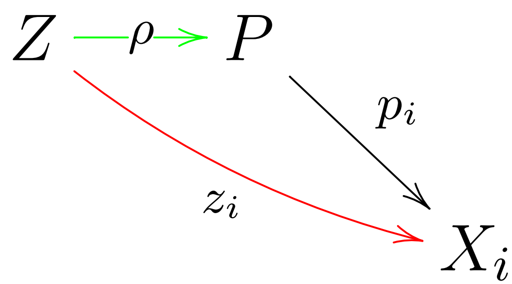
In $\mathbf{SET}$, as in many, many categories, the product is a relatively familiar construction: It is the set of all I-tuples of elements drawn from the sets. More precisely, we have

Lemma

Let $\{X_i\}_{i \in I}$ be a collection of sets. Then, the product is given by $P := \{(x_i)_{i \in I} | x_i \in X_i\}$ and arrows $p_j(x_i)_{i \in I} \mapsto x_j$.

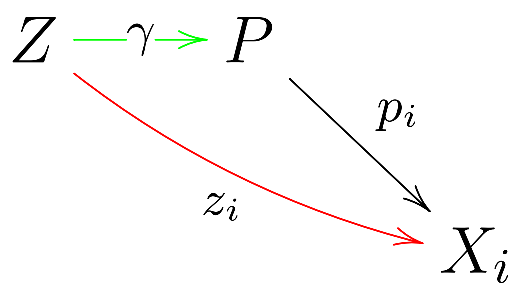
Proof

Let $\{X_i\}$ be a collection of sets. Let Z be a set and $z_i : Z \to X_i$ a collection of maps, one for each $i \in I$. Define $\rho : Z \to P$ by $z \mapsto (z_i(z))_{i \in I}$. Then, *clearly*



commutes for each $i \in I$

Now, suppose that γ is a function so that



commutes. Then, for each $i \in I$, $p_i(\gamma(z)) = z_i(z)$. That is, the i^{th} component of $\gamma(z) = z_i(z)$. This is just the definition of ρ, though. Thus, ρ is unique.

Next Time

Read the section on $limits$.