Quiz

Let $f, g : A \to B$ be two arrows. Define the $equalizer$ of f and g.

Definition

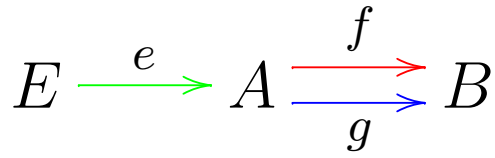
A map e which is an equalizer of two arrow is sometimes called a $regular$ arrow.

Lemma

Every regular arrow is strong.

Proof

Let e be a regular arrow and



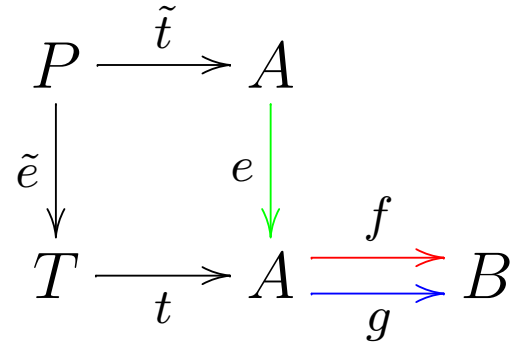
be its equalizer diagram.

Let $t : T \to A$ be an arbitrary arrow and

$$\begin{array}{ccc} P & \xrightarrow{\tilde{t}} & E \\ {\scriptstyle \tilde{e}}\downarrow & & \downarrow{\scriptstyle e} \\ T & \xrightarrow[t]{} & A \end{array}$$

be a pullback diagram. Suppose that $\tilde{e}$ is epic.

Consider now the diagram



$Itcommutes$ (with the exception that $f \neq g$). This means

- $f \circ t \circ \tilde{e} = g \circ t \circ \tilde{e}$

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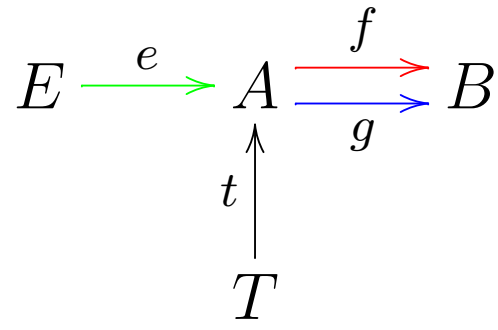
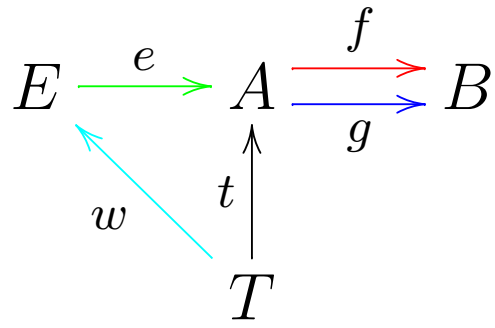
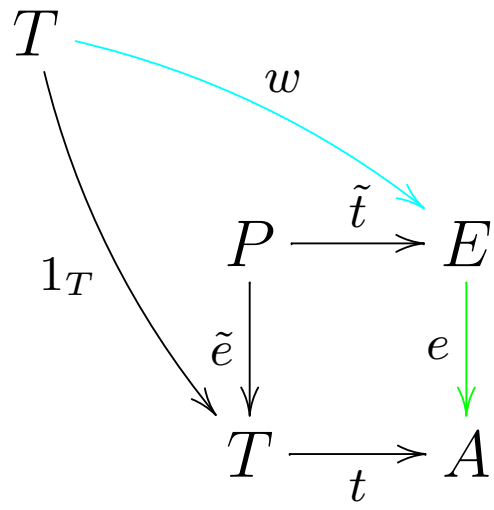


commutes.

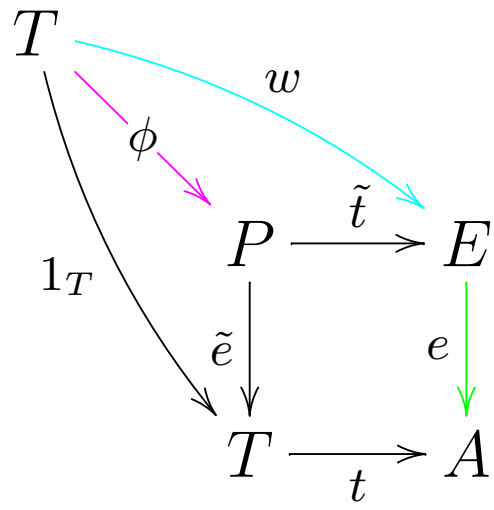
Thus, there exists a unique w so that



commutes. Thus, so too does



Thus, there exists a unique ϕ so that



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- $\phi \circ \tilde{e} = 1_P$ (since $\tilde{e}$, being the pullback of the monic e is itself monic)
- Thus, we have shown that if the pullback of e is epi it must also be an isomorphism. This is the definition of a strong arrow.

Lemma

Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be two functions. Then the equalizer of f and g is the set $E := \{x \in X | f(x) = g(x)\}$ and the function $e : E \to X$ defined by $e(w) = w$ - that is , the inclusion.