

Quiz

Define the **image** of an arrow f in an Abelian category.

Answer:

The image of f is the kernel of the cokernel of f .

Question:

What is the dual construction?

Answer

The **coimage**. It is defined to be the cokernel of the kernel of f .

Last Time:

We proved that in \mathbf{AbG} , the finite product of objects is isomorphic to the finite coproduct. That is, a construction and its dual are essentially the same. In general, this is rare. But not in \mathbf{AbG} .

This Time:

We prove that two other dual constructions are isomorphic in \mathbf{AbG} :

Theorem

In AbG , the image and coimage of an arrow are isomorphic.

We will prove, instead, something more:

Every arrow $f : A \rightarrow B$ in \mathbf{AbG} can be uniquely factored as the composition of a strong epic arrow p followed by a monic arrow i .

Where i is the image and p the coimage.

Proof

Let

$$K \xrightarrow{k} A \xrightarrow[f]{0_{AB}} B$$

be the kernel of f . Note that $f \circ k = 0_B$. Let

$$K \xrightarrow[k]{0_{KA}} A \xrightarrow{p} C$$

be the cokernel of k .

Then,

$$\begin{array}{ccccc} & & B & & \\ & & \uparrow & & \\ & & f & & \\ K & \xrightarrow{k} & A & \xrightarrow{p} & C \\ & \xrightarrow{0_{KA}} & & & \end{array}$$

commutes, which implies that there exists a unique map i so that

$$\begin{array}{ccccc}
 & & B & & \\
 & & \uparrow & \swarrow & \\
 & & f & i & \\
 K & \xrightarrow{k} & A & \xrightarrow{p} & C \\
 & \xrightarrow{0_{KA}} & & &
 \end{array}$$

commutes. In other words, that $f = i \circ p$. Since p is a cokernel, it is a strong epic. Thus, we have left to prove only that i is monic.

So, suppose that $x : X \rightarrow C$ is a map so that $i \circ x = 0$. Let

$$X \begin{array}{c} \xrightarrow{x} \\ \xrightarrow{0_{XC}} \end{array} C \xrightarrow{r} R$$

be the cokernel diagram for x .

Then, **by assumption,**

$$\begin{array}{ccccc} & & B & & \\ & & \uparrow & & \\ & & i & & \\ X & \xrightarrow{x} & C & \xrightarrow{r} & R \\ & \xrightarrow{0_{XC}} & & & \end{array}$$

commutes.

Thus, there exists a unique map q so that

$$\begin{array}{ccccc} & & B & & \\ & & \uparrow i & \swarrow q & \\ X & \xrightarrow{x} & C & \xrightarrow{r} & R \\ & \searrow 0_{XC} & & & \end{array}$$

commutes. Now, since both r and p are epic, so is $r \circ p$. Since we are in an Abelian category, there must exist an arrow $h : H \rightarrow A$ so that

$$H \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{0_{HR}} \end{array} A \xrightarrow{rop} R$$

is a cokernel diagram.

Now,

$$f \circ h = i \circ p \circ h \quad (1)$$

$$= q \circ r \circ p \circ h \quad \text{since } i = q \circ r \quad (2)$$

$$= q \circ 0_{HR} \quad r \circ p \circ h = 0 \quad (3)$$

$$= 0 \quad \text{the composition of} \quad (4)$$

In other words we have the commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{k} & A & \xrightarrow{f} & B \\ & & \uparrow h & \searrow 0_{AB} & \\ & & H & & \end{array}$$

which, of course, implies that there exists a unique map l so that

$$\begin{array}{ccccc}
 K & \xrightarrow{k} & A & \xrightarrow{f} & B \\
 & & \uparrow h & & \\
 & & H & & \\
 & \swarrow l & & &
 \end{array}$$

0_{AB}

commutes. Now,

$$p \circ h = p \circ k \circ l \tag{5}$$

$$= 0_{KC} \circ l \tag{6}$$

$$= 0_{HC} \tag{7}$$

In other words,

$$\begin{array}{ccccc} H & \xrightarrow{h} & A & \xrightarrow{rop} & R \\ & \searrow^{0_{HR}} & \downarrow p & & \\ & & C & & \end{array}$$

commutes. This implies that there exists a unique n so that

$$\begin{array}{ccccc}
 H & \xrightarrow{h} & A & \xrightarrow{r \circ p} & R \\
 & \searrow^{0_{HR}} & \downarrow p & \swarrow n & \\
 & & C & &
 \end{array}$$

commutes. In other words, that $n \circ r \circ p = p = 1_A p$. But remember, that p , being the cokernel of k is epi.

Thus, $n \circ r = 1_A$, which implies that r is monic. But, r , by definition, is a map so that $r \circ x = 0_X$. Since r is monic, we must have that $x = 0$. At the start of all this, though, we let x be an arbitrary map with the property that $i \circ x = 0$. Since this implies that x must also be 0 we see that i is monic. The uniqueness of this factorization is an immediate consequence of a previous lemma.