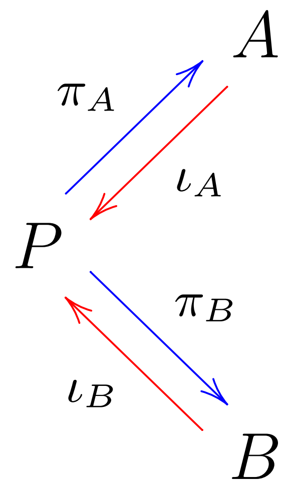
Quiz

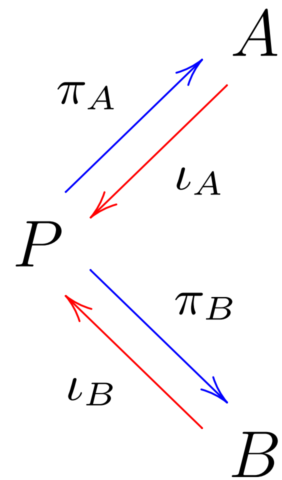
Define zero object

Last Time:

We were proving that in a pre-additive category:The product $A \prod B$ exists if

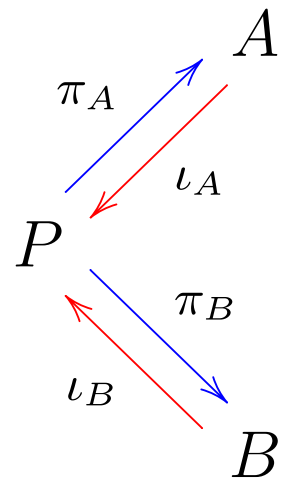


so that



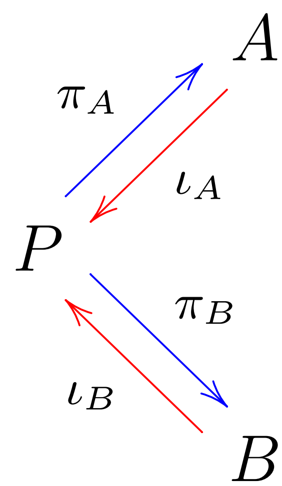
- $\pi_A \circ \iota_A = 1_A$ and $\pi_B \circ \iota_B = 1_B$

so that



- $\pi_A \circ \iota_A = 1_A$ and $\pi_B \circ \iota_B = 1_B$
- $\pi_A \circ \iota_B = 0_{BA}$ and $\pi_B \circ \iota_A = 0_{AB}$

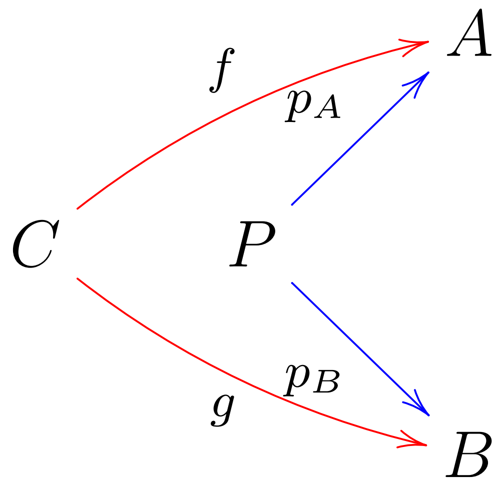
so that



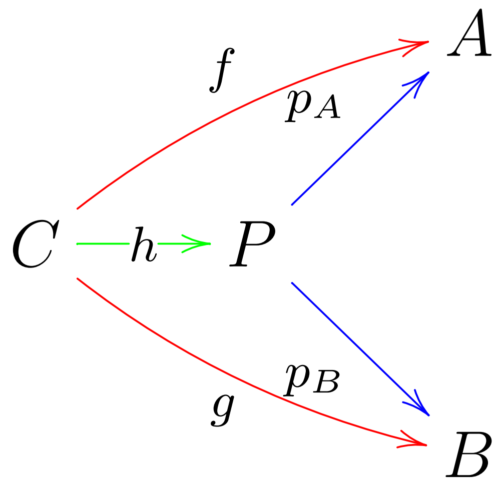
- $\pi_A \circ \iota_A = 1_A$ and $\pi_B \circ \iota_B = 1_B$
- $\pi_A \circ \iota_B = 0_{BA}$ and $\pi_B \circ \iota_A = 0_{AB}$
- $\iota_A \circ \pi_A + \iota_B \circ \pi_B = 1_P$ commutes.

Suppose

C is an object and f and g arrows and consider the diagram:



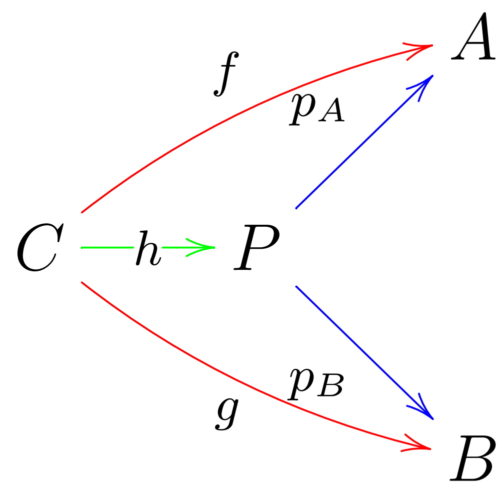
For ease of notation, let us set $h = \iota_A \circ f + \iota_B \circ g$ $Then, we have that$



commutes.

Suppose now

that h' is a map so that



commutes.

Then,

- $h' = 1_P \circ h' = (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \circ h'$

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- $= \iota_A \circ \pi_A \circ h' + d\iota_B \circ \pi_B \circ h'$

Then,

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- $= \iota_A \circ \pi_A \circ h' + d\iota_B \circ \pi_B \circ h'$
- $= \iota_A \circ f + \iota_B \circ g$

Then,

- $h' = 1_P \circ h' = (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \circ h'$
- $= \iota_A \circ \pi_A \circ h' + d\iota_B \circ \pi_B \circ h'$
- $= \iota_A \circ f + \iota_B \circ g$
- $= h$

In other words,

h is unique. Since C, f and g were arbitrary, P and p_A and p_B constitute a product of A and B.

Definition

A product which satisfies (3) in the statement of the above lemma is called a biproduct

definition

A category $\mathbf{C}$ is said to be Abelian if it is additive and, in addition, the following hold:

- Every two objects A and B of $\mathbf{C}$ have a biproduct

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A category $\mathbf{C}$ is said to be Abelian if it is additive and, in addition, the following hold:

- Every two objects A and B of $\mathbf{C}$ have a biproduct
- Every monomorphism is a kernel of some map, and every epimorphism is the cokernel of some map
- Every map has both a kernel and cokernel

lemma

In an Abelian category, the following are equivalent for an arrow $f : A \to B$:

- f is monic

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- f is monic
- $Ker f = 0$

lemma

In an Abelian category, the following are equivalent for an arrow $f : A \to B$:

- f is monic
- $Ker f = 0$
- For all $g : C \to A$, if $f \circ g = 0$, then $g = 0$.

proof

$(1) \Rightarrow (2)$: Consider the diagram

$$0 \xrightarrow{0_A} A \underset{0_{AB}}{\overset{f}{\rightrightarrows}} B$$

It is commutative. Let $d : D \to A$ be a map so that

$$\begin{array}{ccccc} 0 & \xrightarrow{0_A} & A & \underset{0_{AB}}{\overset{f}{\rightrightarrows}} & B \\ & & {\scriptstyle d}\uparrow & & \\ & & D & & \end{array}$$

commutes.

Then

- $f \circ d = 0_{AB} \circ d$

Then

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- $= 0_{DB}$

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- $f \circ d = 0_{AB} \circ d$
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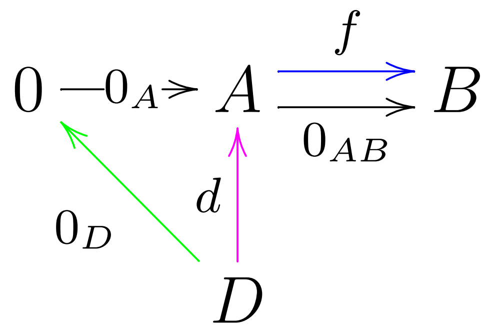
- $= f \circ 0_{DA}$

Then

- $f \circ d = 0_{AB} \circ d$
- $= 0_{DB}$
- $= f \circ 0_{DA}$
- which implies that $d = 0_{DA}$ since f is monic.

Thus,



commutes. Since 0 is terminal, 0_D is the unique such map.

Thus,

$$0 \xrightarrow{0_A} A \underset{0_{AB}}{\overset{f}{\rightrightarrows}} B$$

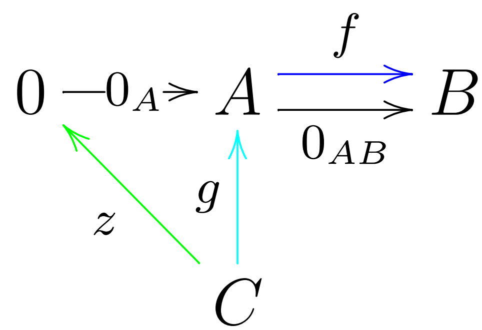
is an equalizer diagram. $(2) \Rightarrow (3)$: Suppose $g : C \to A$ is an arrow and that $f \circ g = 0$. Then, we have the commutative diagram

$$\begin{array}{ccccc} 0 & \xrightarrow{0_A} & A & \underset{0_{AB}}{\overset{f}{\rightrightarrows}} & B \\ & & \uparrow{\scriptstyle g} & & \\ & & C & & \end{array}$$

where the top line is the equalizer of f and 0_{AB}.

Thus,

there exists a uniqe map z so that



commutes.

As before,

however, since the target of z is 0, z must equal 0_C. But this in turn implies that $g = 0_A \circ 0_C = 0_{CA}$.
$(3) \Rightarrow (1)$: Suppose (3). Suppose g and h are arrows from C to A so that

- $f \circ g = f \circ h$ then,

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- $f \circ g = f \circ h$ then,
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- $f \circ g - f \circ h = 0_{CD}$whence,
- $f \circ (g - h) = 0$ so, by assumption
- $g - h = 0_{CD}$ and thus $g = h$ as required

Next Time:

Read §5.3