Quiz

Define what is meant by an *equalizer*
Last Time:

We proved the “only if” part of the
Lemma

A mono $f : A \rightarrow B$ is strong if and only if for every commutative diagram

$$
\begin{array}{ccc}
T & \xrightarrow{t} & A \\
\downarrow{r} & & \downarrow{f} \\
S & \xrightarrow{s} & B
\end{array}
$$

in which $r$ is an epic arrow, there exists a unique arrow $w$
so that in the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{t} & A \\
\downarrow{r} & & \downarrow{f} \\
S & \xrightarrow{s} & B \\
& \searrow{w} & \\
\end{array}
\]

\[f \circ w = s \text{ and } w \circ r = t.\]
This time we will

Complete the proof from last time
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- Prove one or two more facts about strong monos
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- Prove one or two more facts about strong monos
- Define \textit{pre} -- \textit{image}.
This

time, we prove the “if” part:
That is,
Suppose that $f$ is a monic arrow so that for every commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{t} & A \\
\downarrow{r} & & \downarrow{f} \\
S & \xrightarrow{s} & B
\end{array}
\]

in which $r$ is an epic arrow, there exists a unique map $w$
so that in the diagram

\[
\begin{array}{c}
T & \overset{t}{\longrightarrow} & A \\
\downarrow{r} & & \downarrow{f} \\
S & \overset{s}{\longrightarrow} & B \\
\end{array}
\]

\[f \circ w = s \text{ and } w \circ r = t\]
Let $\kappa$ be any arrow and $P \xrightarrow{p} A$ be the pullback of $f$ by $\kappa$. 

be the pullback of $f$ by $\kappa$. 

\[
\begin{array}{c}
P \xrightarrow{p} A \\
q \\
K \xrightarrow{k} B \\
f
\end{array}
\]
Suppose that

$q$ is epic. Then, by hypothesis, there exists a unique $w$ so that

\[ f \circ w = k \text{ and } w \circ q = p. \]
Then, the diagram commutes.
Thus,

there exists a unique arrow $\phi$ so that

\[
\begin{array}{ccc}
K & \xrightarrow{w} & K \\
\phi & \downarrow & \phi \\
1_K & \downarrow & 1_K \\
\end{array}
\]

commutes. Since $f$ is monic, so too is $q$. Since $q \circ \phi = 1_K$, $q$ is an isomorphism.
Lemma

Suppose $f : A \to B$ and that $f = i \circ p$ where $i$ is a strong mono and $p$ is an epi. Suppose too that $f = \bar{i} \circ \bar{p}$. Then, $i = \phi \bar{i}$ and $p = \psi \bar{p}$ where $\phi$ and $\psi$ are isomorphisms.
Proof

Consider, the following commutative diagram

Since $p$ is a strong epi, and $\tilde{i}$ is monic, there exists an unique arrow $u$ so that in the diagram
$u \circ \tilde{p} = p$ and $i \circ u = \tilde{i}$. Since we have assumed that $\tilde{p}$ is a strong epi, there exists a unique $v$ so that in the diagram
$v \circ p = \tilde{p}$ and $\tilde{i} \circ v = i$. 
Now,

\[ i \circ u \circ v \circ p = \tilde{i} \circ \tilde{p} \]
Now,

\[ i \circ u \circ v \circ p = \tilde{i} \circ \tilde{p} \]

\[ = i \circ p \Rightarrow \]
Now,

- $i \circ u \circ v \circ p = \tilde{i} \circ \tilde{p}$
- $= i \circ p \implies$
- $u \circ v \circ p = p$ (since $i$ is monic)
Now,

- $i \circ u \circ v \circ p = \tilde{i} \circ \tilde{p}$
- $= i \circ p \Rightarrow$
- $u \circ v \circ p = p$ (since $i$ is monic)
- $u \circ v = 1_K \circ p$ (which implies)
Now,

\[ i \circ u \circ v \circ p = \tilde{i} \circ \tilde{p} \]
\[ = i \circ p \Rightarrow \]
\[ u \circ v \circ p = p \text{ (since } i \text{ is monic)} \]
\[ u \circ v = 1_K \circ p \text{ (which implies)} \]
\[ u \circ v = 1_K \text{ (since } p \text{ is epic)} \]
Now,

\[ i \circ u \circ v \circ p = \tilde{i} \circ \tilde{p} \]

\[ = i \circ p \Rightarrow \]

\[ u \circ v \circ p = p \text{(since } i \text{ is monic)} \]

\[ u \circ v = 1_K \circ p \text{ (which implies)} \]

\[ u \circ v = 1_K \text{(since } p \text{ is epic)} \]
Similarly we can show that $v \circ u = 1_G$.

Thus, both $v$ and $u$ are isomorphisms. In particular, $i$ and $p$ are isomorphic to $\tilde{i}$ and $\tilde{p}$ respectively.
Lemma

Let $f : X \to Y$ be a function of sets and $A \subseteq Y$. If $a : A \to Y$ is the inclusion function, that is the function which takes each element of $A$ to itself, then the pullback of $f$ and $a$ is the pre-image of $A$ by $f$, $f^{-1}(A)$. 
Proof

Let’s set up the diagram first:

\[
\begin{array}{ccc}
  f^{-1} & i & \rightarrow & X \\
  \bar{f} & \downarrow & & \downarrow f \\
  A & a & \rightarrow & Y \\
\end{array}
\]

Here, \( i \) is the inclusion function, and \( \bar{f} \) is the restriction of \( f \) to \( f^{-1}(A) \).
Now, let $Z$ be a set and $g$ and $h$ functions so that

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
\downarrow{h} & & \downarrow{f} \\
A & \xrightarrow{a} & Y \\
\end{array}
\]

commutes. That is, for every $x \in H$, $f(g(x)) = a(h(x))$. 
This allows us to say then that the diagram
\[
\begin{array}{c}
\text{This}
\end{array}
\]

allows us to say then that the diagram

\[
\begin{array}{ccccccccc}
Z & \xrightarrow{g} & X \\
\downarrow{\phi} & & \downarrow{f} \\
A & \xrightarrow{\phi^{-1}} & X
\end{array}
\]

commutes, where \( \phi(x) := g(x) \).
This definition makes sense because \( f(g(x)) = h(x) \), and \( \bar{f}\phi(x) = f(g(x)) \). If \( \psi \) were another map which made commute, then it would also follow that \( i\phi = i\psi \) which implies that \( \phi = \psi \) since \( i \) is one to one.
Definition

In any category with pullbacks, if \( \iota H \to G \) is monic and \( f : K \to G \) is an arrow, we will denote the pullback of \( f \) by \( \iota 
\)

\[ f^{-1}(H) \]

and refer to it as \textit{pre-image} of \( H \) by \( f \).
For next time:

Read the section on equalizers