Quiz

Define what is meant by an $equalizer$

Last Time:

We proved the “only if” part of the

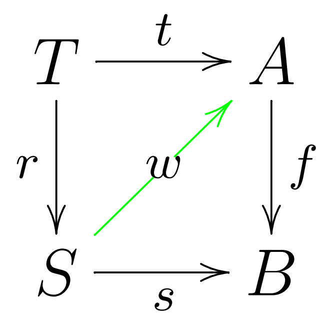
Lemma

A mono $f : A \to B$ is strong if and only if for every commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

in which r is an epic arrow, there exists a unique arrow w

so that in the diagram



$f \circ w = s$ and $w \circ r = t$.

This time we will

- Complete the proof from last time

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- Define $pre - image$.

This

time, we prove the “if” part:
That is,

Suppose that f is a monic arrow so that for every commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

in which r is an epic arrow, there exists a unique map w

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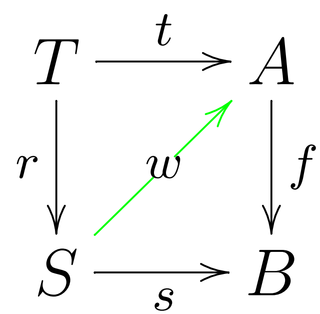


$f \circ w = s$ and $w \circ r = t$

Let k be any arrow

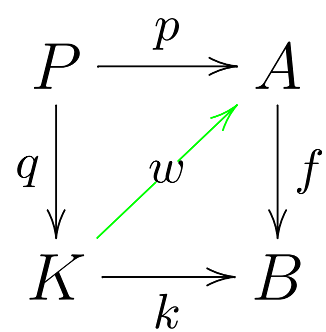
and

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ K & \xrightarrow[k]{} & B \end{array}$$

be the pullback of f by k.

Suppose that

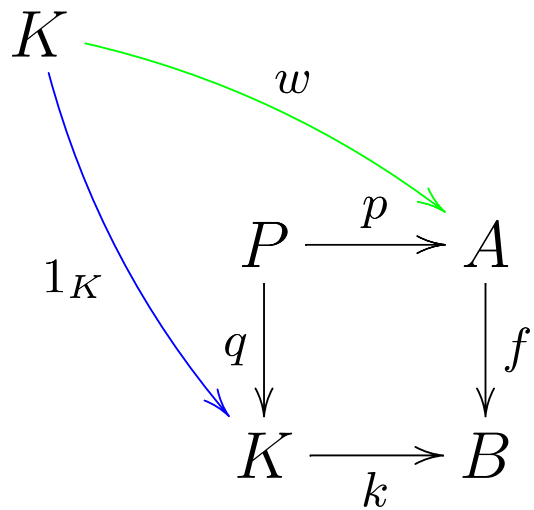
q is epic. Then, by hypothesis, there exists a unique w so that



$f \circ w = k$ and $w \circ q = p$.

Then,

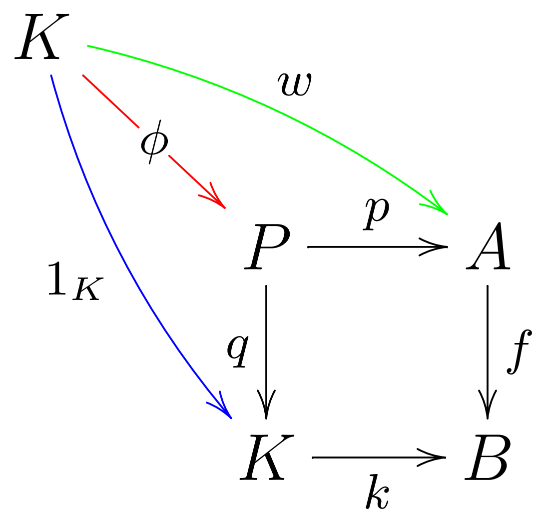
the diagram



commutes.

Thus,

there exists a unique arrow ϕ so that



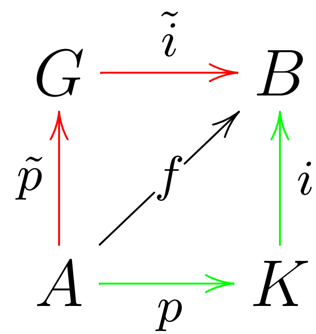
commutes. Since f is monic, so too is q. Since $q \circ \phi = 1_K$, q is an isomorphism.

Lemma

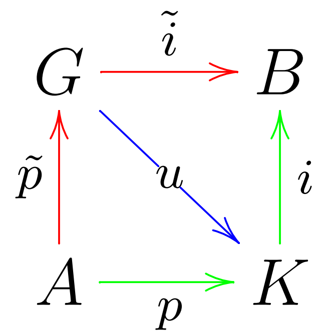
Suppose $f : A \to B$ and that $f = i \circ p$ where i is a strong mono and p fis an epi. Suppose too that $f = \bar{i} \circ \bar{p}$. Then, $i = \phi\bar{i}$ and $p = \psi\bar{p}$ where ϕ and ψ are isomorphisms.

Proof

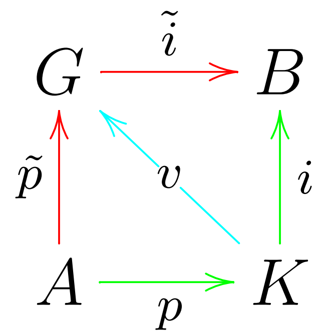
Consider, the following commutative diagram



Since p is a strong epi, and $\tilde{i}$ is monic, there exists an unique arrow u so that in the diagram



$u \circ \tilde{p} = p$ and $i \circ u = \tilde{i}$. Since we have assumed that $\tilde{p}$ is a strong epi, there exists a unique v so that in the diagram



$v \circ p = \tilde{p}$ and $\tilde{i} \circ v = i$.

Now,

- $i \circ u \circ v \circ p = \tilde{i} \circ \tilde{p}$

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- $u \circ v = 1_K \circ p$ (which implies)

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$*$ Similarly we can show that $v \circ u = 1_G$.
Thus, both v and u are isomorphisms. In particular, i and p are isomorphic to $\tilde{i}$ and $\tilde{p}$ respectively.

Lemma

Let $f : X \longrightarrow Y$ be a function of sets and $A \subseteq Y$. If $a : A \longrightarrow Y$ is the inclusion function, that is the function which takes each element of A to itself, then the pullback of f and a is the pre-image of A by f, $f^{-1}(A)$.

Proof

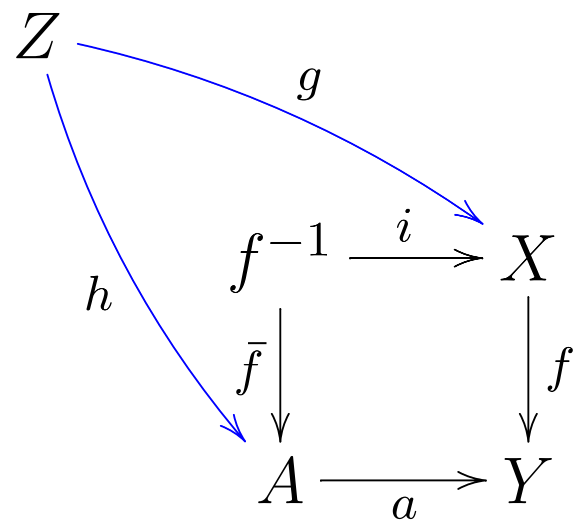
Let's set up the diagram first:

$$\begin{array}{ccc} f^{-1} & \xrightarrow{i} & X \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[a]{} & Y \end{array}$$

Here, i is the inclusion function, and $\bar{f}$ is the restriction of f to $f^{-1}(A)$.

Now,

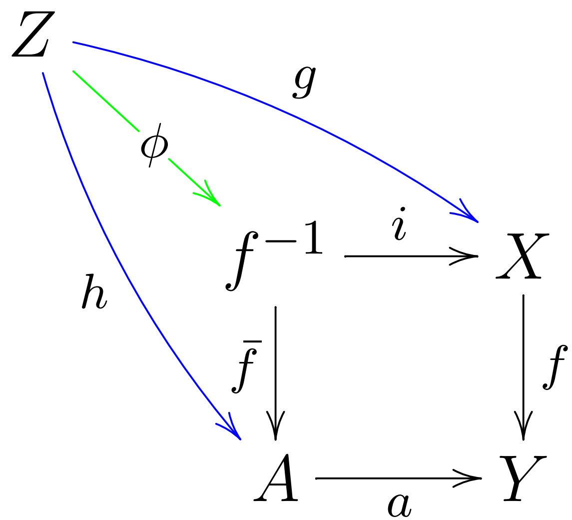
let Z be a set and g and h functions so that



commutes. That is, for every $x \in H$, $f(g(x)) = a(h(x))$.

This

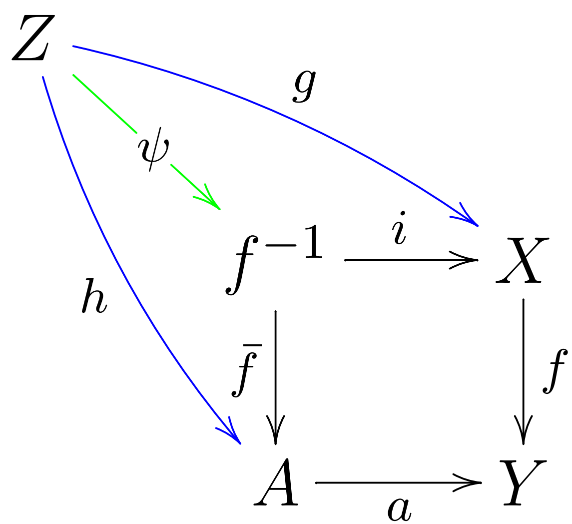
allows us to say then that the diagram



commutes, where $\phi(x) := g(x)$.

This

definition makes sense because $f(g(x)) = h(x)$, and $\bar{f}\phi(x) = f(g(x))$. If ψ were another map which made



commute, then it would also follow that $i\phi = i\psi$ which implies that $\phi = \psi$ since i is one to one.

Defintion

In any category with pullbacks, if $\iota H \longrightarrow G$ is monic and $f : K \longrightarrow G$ is an arrow, we will denote the pullback of f by ι $f^{-1}(H)$ and refer to it as $pre - image$ of H by f.

For next time:

Read the section on equalizers