Define the *product* of two objects $A$ and $B$. 
Last Time

- Construction of equalizers in SET
Last Time

- Construction of equalizers in SET
- Relationship between “regular” and “strong”
This Time

Definition of Product
This Time

- Definition of Product
- Construction of Products in \textbf{SET}
This Time

- Definition of Product
- Construction of Products in $\text{SET}$
- Definition of Limits
This Time

- Definition of Product
- Construction of Products in $\text{SET}$
- Definition of Limits
- Limits as “all”
Definition

Let \( \{A_i\}_{i \in I} \) be a collection of objects. The product of these objects, if it exists, is defined to be an object \( P \) and arrows \( \{p_i : P \rightarrow A_i\}_{i \in I} \) with the following property: For any object \( D \) and arrows \( \{d_i : D \rightarrow A_i\} \) there is a unique map \( \rho \) so that for every \( i \in I \)

\[
\begin{array}{ccc}
D & \xrightarrow{\rho} & P \\
\downarrow{d_i} & & \downarrow{p_i} \\
& A_i &
\end{array}
\]

commutes.
In SET, as in many, many categories, the product is a relatively familiar construction: It is the set of all $I$-tuples of elements drawn from the sets. More precisely, we have
Lemma

Let \( \{X_i\}_{i \in I} \) be a collection of sets. Then, the product is given by \( P := \{(x_i)_{i \in I} | x_i \in X_i\} \) and arrows \( p_j((x_i)_{i \in I}) \mapsto x_j \).
Proof

Let \( \{X_i\} \) be a collection of sets. Let \( Z \) be a set and \( z_i : Z \to X_i \) a collection of maps, one for each \( i \in I \). Define \( \rho : Z \to P \) by \( z \mapsto (z_i(z))_{i \in I} \). Then, clearly

\[
\begin{array}{ccc}
Z & \xrightarrow{\rho} & P \\
\downarrow{z_i} & & \downarrow{p_i} \\
X_i & & 
\end{array}
\]

commutes for each \( i \in I \).
Now, suppose that $\gamma$ is a function so that $Z \xrightarrow{\gamma} P$ commutes. Then, for each $i \in I$, $p_i(\gamma(z)) = z_i(z)$. That is, the $i^{th}$ component of $\gamma(z) = z_i(z)$. This is just the definition of $\rho$, though. Thus, $\rho$ is unique.
We end the introduction to categories with the grand daddy of all of these universal objects:
Limits
Definition

Let \( \{A_i\}_{i \in I} \) be a collection of objects and \( \{f_{jk} : A_j \to A_k\}_{j, k \in I} \) a collection of arrows between them.
Then, the *limit* of these collections, if it exists, is an object $L$ and arrows $\{l_i\}_{i \in I}$, one for each $i \in I$ so that for every $f_{jk}$ the diagram

![Diagram](https://via.placeholder.com/150)

commutes
and, given any other object $T$ and arrows $\{t_i\}_{i \in I}$ so that

\[
\begin{array}{c}
A_j \\
| \\
\downarrow t_j \\
T \\
| \\
\downarrow f_{jk} \\
A_k \\
| \\
\downarrow t_k
\end{array}
\]

commutes for every $j, k$, 
there exists a unique map $\lambda$ so that

commutes.
Question:

How is this a generalization of the other constructions we’ve defined in this chapter?
Answer

Exercise
But,

Let’s take a look at one of them:
Note that the product is a limit in the case that the set of arrows between the given objects $\{A_i\}_{i \in I}$ is empty!
Next Time:

Read pages 31-35 in the text.