Quiz

Define the $product$ of two objects A and B.

Last Time

- Construction of equalizers in $\mathbf{SET}$

Last Time

- Construction of equalizers in $\mathbf{SET}$
- Relationship between “regular” and “strong”

This Time

- Definition of Product

This Time

- Definition of Product
- Construction of Products in $\mathbf{SET}$

This Time

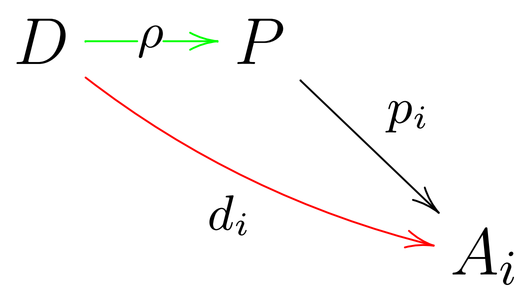
- Definition of Product
- Construction of Products in $\mathbf{SET}$
- Definition of Limits

This Time

- Definition of Product
- Construction of Products in $\mathbf{SET}$
- Definition of Limits
- Limits as “all”

Definition

Let $\{A_i\}_{i\in I}$ be a collection of objects. The $\color{green}{product}$ of these objects, if it exists, is defined to be an object P and arrows $\{p_i : P \longrightarrow A_i\}_{i\in I}$ with the following property: For any object D and arrows $\{d_i : D \longrightarrow A_i\}$ there is a unique map ρ so that for every $i \in I$



commutes.

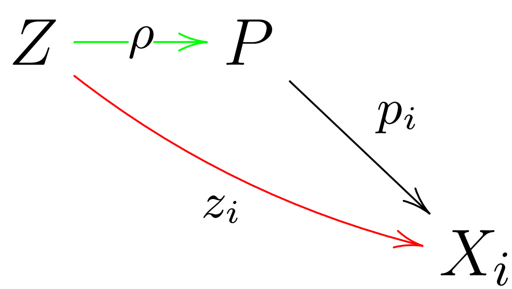
In $\mathbf{SET}$, as in many, many categories, the product is a relatively familiar construction: It is the set of all I-tuples of elements drawn from the sets. More precisely, we have

Lemma

Let $\{X_i\}_{i \in I}$ be a collection of sets. Then, the product is given by $P := \{(x_i)_{i \in I} | x_i \in X_i\}$ and arrows $p_j (x_i)_{i \in I} \mapsto x_j$.

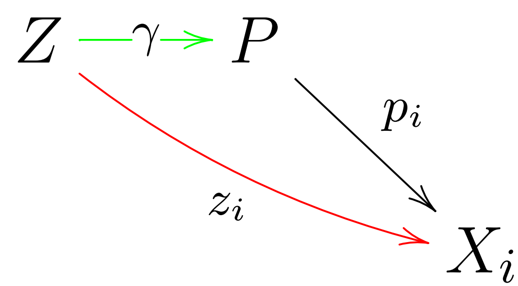
Proof

Let $\{X_i\}$ be a collection of sets. Let Z be a set and $z_i : Z \longrightarrow X_i$ a collection of maps, one for each $i \in I$. Define $\rho : Z \longrightarrow P$ by $z \mapsto (z_i(z))_{i \in I}$. Then, $clearly$



$commutes\ for\ each\ i \in I$

Now, suppose that γ is a function so that



commutes. Then, for each $i \in I$, $p_i(\gamma(z)) = z_i(z)$. That is, the i^{th} component of $\gamma(z) = z_i(z)$. This is just the definition of ρ, though. Thus, ρ is unique.

Limits

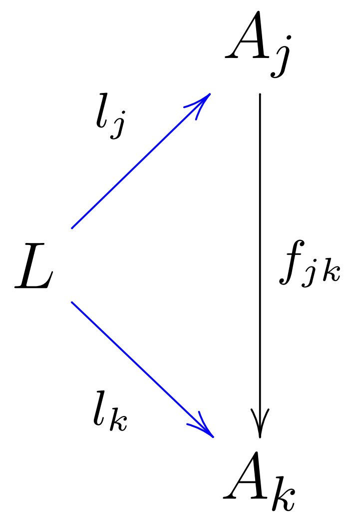
We end the introduction to categories with the grand daddy of all of these universal objects:

Limits

Definition

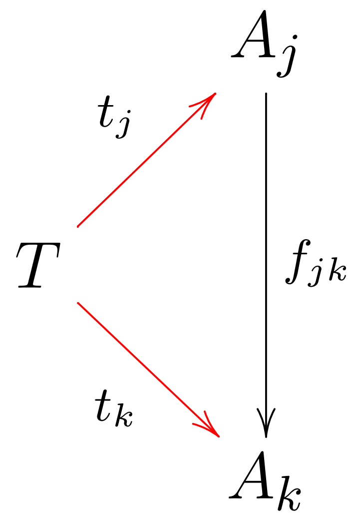
Let $\{A_i\}_{i\in I}$ be a collection of objects and
$\{f_{jk} : A_j \rightarrow A_k\}_{j,k\in I}$ a collection of arrows between them.

Then, the *$limit$* of these collections, if it exists, is an object L and arrows $\{l_i\}_{i \in I}$, one for each $i \in I$ so that for every f_{jk} the diagram



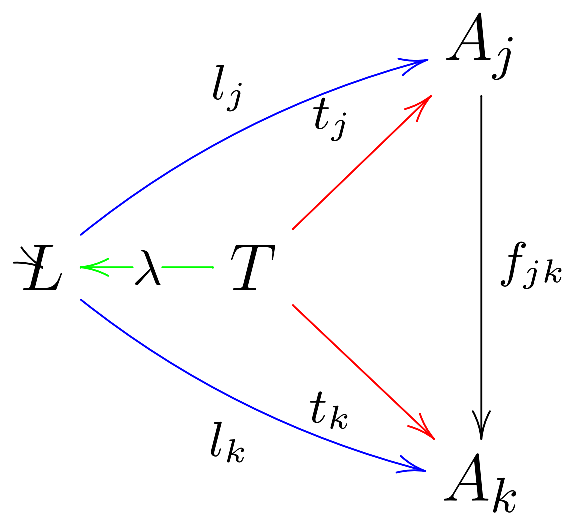
commutes

and, given any other object T and arrows $\{t_i\}_{i \in I}$ so that



commutes for every j, k,

there exists a unique map λ so that



commutes.

Question:

How is this a generalization of the other constructions we've defined in this chapter?

Answer

Exercise

But,

Let's take a look at one of them:

Product

Note that the product is a limit in the case that the set of arrows between the given objects$\{A_i\}_{i \in I}$ is empty!

Next Time:

Read pages 31-35 in the text.