(Really) Modern Algebra

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Chapter 1

Introduction

Many books on modern algebra introduce category theory only as an afterthought. This book will take a different approach. Here we seek to introduce the typical subjects of such a book: Groups, Rings, Modules, and Fields in so far as possible, through the lens of category theory. Category theory has played an important role in the development of many mathematical disciplines for seven decades now. In particular, algebraic geometry, which provided the motivation for the development of so much commutative ring theory - a major topic of this book - relies on many of the notions and facts of category theory. We will attempt to understand groups, rings modules and fields by first understanding the categories in which they live, and then, as necessary, understanding the internal structure of the objects themselves. Any graduate student intent on specializing in algebra or topology will at some time encounter categorical ideas. It is my assertion that treating categories as the basic notion of study presents no greater barrier to understanding modern mathematics than does treating sets as the basic notion of study. What's more, I believe that for those who study subjects such as algebra it presents less of a barier.

Chapter 2

Partially Ordered Sets

As preparation for much of the material to be studied a few chapters from now and as a pool from which we can draw examples of objects defined in the chapters nearer at hand we (re-?)introduce some of the basic definitions and facts about partial orders on sets.

Definition 2.0.1. *Let P be a set. A partial ordering on P is a relation usually denoted $\leq$ which satisfies the following criteria for all $x, y, z \in P$.*

1. *$x \leq x$ (Reflexivity)*
2. *$x \leq y$ and $y \leq z \Rightarrow x \leq z$ (Transitivity)*
3. *$x \leq y$ and $y \leq x \Rightarrow x = y$ (Anti-Symmetry)*

We will often denote a partially ordered set as that above as an ordered pair $(P, \leq)$.

Examples of partially ordered sets litter the mathematical landscape. The Real numbers $\mathbb{R}$ with the usual less than or equal to relation satisfies the above definition. So do all of the subsets of $\mathbb{R}$: $\mathbb{Q}$ (the rationals), $\mathbb{Z}$ (the integers) etc. If $\mathbb{Z}[X]$ is the collection of all polynomials with integer coefficients, the relation $p \leq q$ if and only if the degree of p is less than or equal to the degree of q defines a partial order. So too does the subset relation, $\subseteq$ on the power set of a set X, $\mathcal{P}(X)$. (Recall that given a set X, we define its power set $\mathcal{P}(X) := \{A | A \subseteq X\}$). Indeed,

Lemma 2.0.2. *Let X be a set, and $\mathcal{P}(X)$ its power set. Then, $\subseteq$ defines a partial order on $\mathcal{P}(X)$.*

Proof. Let A, B, and C be subsets of X.

1. $A \subseteq A$

2. Suppose that $A \subseteq B$ and $B \subseteq C$. Let $x \in A$ be arbitrary. Then $x \in B$ and, thus, $x \in C$. Since x was arbitrary, we have that $A \subseteq C$.

3. Suppose $A \subseteq B$ and $B \subseteq A$. Then, by definition, $A = B$.

□

One of the beautiful things about mathematics is that it does not limit its practitioners to examples which occur only in nature. For example one could, at her whim, simply make up a set and a partial ordering of that set. Say $X := \{a, b, c\}$ and let $a \leq a$, $b \leq b$ and $c \leq c$ be the only relations among the elements of X defined by $\leq$. This is clearly a partial order. It is sometimes referred to as the trivial partial order.

Exercise 2.0.3. *Show that if we add the relations $a \leq b$ and $c \leq b$, then we still have a partial order.*

Exercise 2.0.4. *Does the relation "A is a strict subset of B", $A \subsetneq B$ define a partial order on $\mathcal{P}(X)$?*

Exercise 2.0.5. *Let A be the set of all English words. For any two words $x, y \in A$, define $x \leq y$ if and only if the letters of x appear consecutively and in the correct order in the word y. For example, if x is the word "to" and y is the word "toward" then $x \leq y$. On the other hand if z is the word "trout" then $x \nleq z$. Similarly if w is the word "other", than $x \nleq w$. Show that in this case $\leq$ is a partial order on A.*

Exercise 2.0.6. *Is the relation $x \leq y$ if and only if x divides y a partial order on the integers $\mathbb{Z}$?*

Definition 2.0.7. *A partial order $\leq$ on a set X is said to be a total order if for ever $x, y \in X$, either $x \leq y$ or $y \leq x$. A set with a total order will be said to be totally ordered.*

Exercise 2.0.8. *Give an example of a totally ordered set. Give an example of a set with a partial order which is not a total order.*

Definition 2.0.9. *Let $(P, \leq)$ be a partially ordered set. Let $Q \subseteq P$. The least upper bound of Q, if it exists, is that element $u \in P$ so that*

1. *$q \leq u$ for all $q \in Q$.*

2. *If $q \leq s$ for all $q \in Q$, then $u \leq s$.*

We also have the dual notion:

Definition 2.0.10. *Let $(P, \leq)$ be a partially ordered set. Let $Q \subseteq P$. The lower bound of Q, if it exists, is that element $u \in P$ so that*
$u \leq q$ for all $q \in Q$.
If, in addition, b also satisfies
If $b \leq q$ for all $q \in Q$, then $b \leq u$.
Then we shall say that b is the greatest lower bound of Q.

If the entire partially ordered set has a least upper bound or a greatest lower bound, these are referred to as the top and bottom elements respectively.

Exercise 2.0.11. *Prove that every finite subset of a partially ordered set must have a greatest lower and least upper bound.*

Exercise 2.0.12. *Give an example of a partially ordered set in which every subset has a least upper bound, but which also contains subsets with no greatest lower bound.*

Exercise 2.0.13. *Construct a partially ordered set which has a top and bottom element, but which also contains subsets with no greatest lower bound and subsets with no least upper bound.*

Exercise 2.0.14. *Give an example of a partially ordered set with a top element but no bottom element. Also give an example of a partially ordered set with a bottom element but no top element.*

Exercise 2.0.15. *A partially ordered set in which every subset has a least upper bound is said to have the Least Upper bound Property. Show that the rational numbers $\mathbb{Q}$ do not have the least upper bound property.*

Exercise 2.0.16. *Show that the least upper bound is unique.*

Some partially ordered sets which lack top or bottom elements have elements which are somewhat like top or bottom elements.

Definition 2.0.17. *An element m of a partially ordered set $(X, \leq)$ is called maximal (respectively minimal) if for all $x \neq m \in X$, $m \nleq x$ (resp. $x \nleq m$).*

Exercise 2.0.18. *Show that every top element is also maximal.*

Exercise 2.0.19. *Give an example of a set wih a maximal element which is not a top element.*

Exercise 2.0.20. *Give an example of a partially ordered set with distinct maximal elements.*

Exercise 2.0.21. *Show that in a totally ordered set, every maximal element is also a top element.*

Definition 2.0.22. *A partially ordered set in which every pair of elements has a least upper bound is called a lattice.*

Lemma 2.0.23. *Let X be a set and $\mathcal{P}(X)$ its power set. Then $(\mathcal{P}(X), \subseteq)$ is a lattice.*

Proof. We have already seen that $(\mathcal{P}(X), \subseteq)$ is a partiall ordered set, so we have only left to prove that any two subsets $A \subseteq X$ and $B \subseteq X$ possess both a greatest lower and least upper bound. Let's handle the least upper bound first:

Clearly, the relations $A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$ hold. Suppose now that both $A \subseteq C$ and $B \subseteq C$ also hold for some $C \subseteq X$. Let $x \in A \bigcup B$. Then, by definition of $\bigcup$, $x \in A$. Since $A \subseteq C$, $x \in C$. Similarly $x \in A \bigcup B$ implies that $x \in B$ and so $x \in C$ since $B \subseteq C$. Thus, $A \bigcup B \subseteq C$. Since C was an arbitrary set containing both A and B we can conclude that every set containing A and B also contains $A \bigcup B$. Thus, by definition, $A \bigcup B$ is the least upper bound of A and B in $(\mathcal{P}, \subseteq)$. Similarly, $A \bigcap B$ is the greatest lower bound of A and B □

Exercise 2.0.24. *Prove the statement in red at the end of the proof immeadiately above.*

Exercise 2.0.25. *Show that $\mathbb{R}$ with the usual order relation is a lattice.*

Exercise 2.0.26. *Show that every totally ordered set is a lattice.*

Exercise 2.0.27. *Show that in a lattice any finite collection of elements has both a least upper bound and a greater lower bound.*

In coming sections we will meet other important lattices. For now, let us establish a notational convention which will prove useful then. If $\{x_i\}_{i=1}^n$ are n elements of a partially ordered set X, then we will denote their greatest lower bound $\wedge_{i=1}^n x_i$ and least upper bound $\vee_{i=1}^n x_i$.

Chapter 3

A Note About Foundations

Category theory has as its main concern collections of mathematical objects grouped by species and maps which preserve the essential structure which defines that particular species. Thus, we will often use phrases such as "the collection of all X" where X stands for one of these types of mathematical objects. This - as has been known for quite some time - leads to trouble. In fact, shortly after Frege completed his volume the surely uncontroversial Basic Laws of Arithmetic, he recieved a note from Bertrand Russell who pointed out that one of Frege's axioms contained within it a contradiction. The problem comes from using phrases such as "the collection of all ...". The example of this which has since become known as "Russell's paradox" defines the following set: $T := \{S|S \notin S$ where S is any set. The question is: Is $T \in T$? According to the axioms of set theory, as well as the axioms laid out by Frege, no error has been committed in construction so far. But there are only two possibities:

1. $T \in T$
2. $T \notin T$.

However, if $T \in T$, then, by definition of T, $T \notin T$; a contradiction. On the other hand, if $T \notin T$, then, by definition of T, we must have that $T \in T$; again a contradiction. What are left to conclude, then? The axioms or rules which allowed us to write down this statement must be somehow flawed. Certainly, though, we do not want to abandon the notion of element in a set, or defining a set according to some property. It turns out that it is better to dispense with the idea that everything we can define must be a set. In fact one of the most popular preventitives against the sort of situation Russell pointed out to Frege was proposed by Grothendieck. His idea was to always work within a collection called "the universe". We declare everything within that universe to be a set, but not the universe itself. With this approach internal contradictions like the paradox above are avoided.

Chapter 4

Category Theory

We start by defining a category:

A Category consists of a collection $A, B, C, ...$ of objects and for each pair of objects A, B a set $Hom(A, B)$ of arrows or morphisms so that

1. For any three objects A, B, C a composition law

$$Hom(A, B) \times Hom(B, C) \to Hom(A, C)$$

   which assigns to any pair of arrows $f \in Hom(A, B)$ and $g \in Hom(B, C)$ an element $g \circ f \in Hom(A, C)$ the composite of f with g. If such an assignment exists we will say that f and g are composable.

2. The operation above, referred to as composition, is associative. That is for composable arrows f,g, and h we have that$f \circ (g \circ h) = (f \circ g) \circ h$.

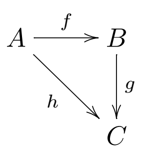
3. For all objects A there exists an element $1_A \in Hom(A, B)$ called the identity of A with the property that for any arrow f with which it is composable the composition of it with f is equal to f.

We will often depict an arrow $f \in Hom(A, B)$ as literally an arrow between A and B. That is

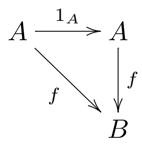
$$A \xrightarrow{f} B$$

Extending the metaphor in this situation we will often refer to A as the source of f and B as the target. A good example of a category is the collection of sets and functions between them.

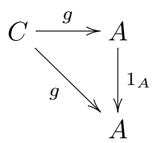
In fact, this sort of depiction is one of the many benefits of "thinking categorically." One can often see at a glance some relationship which would otherwise require long strings of logical quantifiers and equations. For example, we can (and often will) depict the fact that $h = g \circ f$ with the diagram



That is, taking the path from A to C along f and g is exactly the same - equal - as taking the path from A to C along h. At times we will consider diagrams like that above in which the only assumption is that the diagram give an accurate account of the source and target of the named arrows. That is, we may not necessarily suppose that $h = g \circ f$. If $h = g \circ f$, we will say that the diagram commutes. For example, we would depict the nature of the identity element by giving the diagrams



and



Exercise 4.0.28. *Give two more examples of categories. Remember that you must specify both the objects (e.g. "sets") and the arrows(e.g."functions"). Keep in mind too that the definition above requires that the composition of any two arrows in the category you've defined must meet the definition of arrow for your category as well.*

Speaking of which

Exercise 4.0.29. *Give two examples of functions f, g so that both have a property P, but $g \circ f$ does not.*

The properties of algebraic categories which we will examine can be completely characterized by the sorts of arrows which exist and properties that the arrows may or may not have. For example:

An arrow $f : A \to B$ is said to be monic or a mono if, whenever $f \circ g = f \circ h$, it must be that $g = h$.
An arrow $f : A \to B$ is said to be epic or a epi if, whenever $h \circ f = g \circ f$, it must be that $g = h$.

Exercise 4.0.30. *Show that the composite of two monics is again a monic and that the compositie of two epics is again an epic.*

In the category whose objects are sets and arrows the functions between them it turns out that an arrow is monic if and only if it is 1-1, and epi if and only if it is onto. Indeed:

Exercise 4.0.31. *Prove that a function between sets is monic if and only if it is 1-1 and epi if and only if it is onto.*

Lemma 4.0.32. *Suppose that $h = g \circ f$ is monic. Then so too is f.*

Proof. Suppose m and n are maps so that $f \circ m = f \circ n$. Then $g \circ f \circ m = g \circ f \circ n$. That is $h \circ m = h \circ n$. But h is monic by assumption. Thus, $m = n$. Since m and n were arbitrary, f must be monic. □

Exercise 4.0.33. *Prove the dual of* **??**.

Later we shall meet epis which are not onto. Another important kind of arrow which we shall consider approximates the behavior of the indentity arrow. An isomorphism $f : A \to B$ is an arrow for which there exists an arrow $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. In such a case, g may be referred to as the inverse of f. If there exists an isomorphism $f : A \to B$ then we shall say that A and B are isomorphic. We will denote this by writing $A \simeq B$. If there exists an isomorphism ϕ so that $h \circ \phi = k$, then we shall say that h and k are isomorphic.

Exercise 4.0.34. *Prove that the composition of two isomorphisms is again an isomorphism.*

Exercise 4.0.35. *Prove that in any category, for every object A in that category, there exists an isomorphism from $A \to A$. Name it and give its inverse.*

Exercise 4.0.36. *Prove that an isomorphism is both epi and monic.*

We will see soon enough that the converse of this exercise does not hold in general. There is an important instance in which it does,however.

Exercise 4.0.37. *Show that the inverse of an isomorphism is unique.*

Exercise 4.0.38. *Prove that the converse holds in the category of sets and functions. That is, prove that if a function between sets is both monic and epi, then it is an isomorphism.*

From now on we will denote the category of sets, both its objects and arrows, as **SET**.

So-called universal diagrams and universal objects play a fundamental role. Their existence or non-existence distinguishes one sort of category from another.

A initial object in a category **C** is an object I so that for every other object A of **C** the set $Hom(I, A)$ consists of exactly one element.

Turning things around, we have the notion of a terminal object:

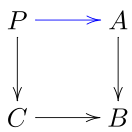
A terminal object in a category **C** is an object T so that for every object B of **C** the set $Hom(B, T)$ consists of exactly one element.

Exercise 4.0.39. *Prove that any two initial or terminal objects are isomorphic.*

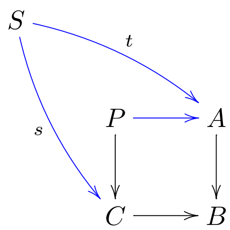
Exercise 4.0.40. *Does* **SET** *have an initial object? terminal object? What are they?*

We shall soon meet a category in which the initial and terminal objects are one and the same. When such happens we refer to this object (and its isomorphic copies) as a zero object, and denote it **0**.
Universal diagrams are those diagrams in which there is some unique arrow which, if added to the original diagram, produces a new commutative diagram. They, like initial and terminal objects, are unique "up to isomorphism". More precisely, any two such objects are isomorphic and any two corresponding maps in such diagrams are the same up to - possibly - composition with an isomorphism. This is one unfortunate price we will pay by adhering to the categorical viewpoint. We may only be able to make a valid claim about a group of isomorphic objects or arrows. We may only be able to say that two objects are isomorphic as opposed to saying that they are equal.
Nonetheless, examples of these universal objects abound and play a fundamental role. Our first example is the pullback, or fibreproduct.

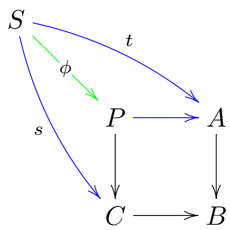
Let $f : A \to B$ and $g : C \to B$ be arrows in a category **C**. The pullback of f by g, or symmetrically, of g by f is a commutative diagram



so that whenever there exits an object S and arrows s and t, so that the diagram



commutes, there exists a unique ϕ so that



commutes.
Before we go any further, it will profit us greatly to consider the notion of "duality":

Let **C** be a category. The dual category of **C**, denoted $\mathbf{C}^*$, is that category whose objects are precisely the objects of **C** and whose arrows are precisely the arrows of **C**, but which point in the other direction. That is for each arrow $f : A \to B$ of **C** there is exactly one arrow $f^* : B \to A$ of $\mathbf{C}^*$, and for each $g^* : C \to D$ in $\mathbf{C}^*$ there is exactly one corresponding arrow $g : D \to C$ in **C**. Note that this implies that if $g \circ f = h$ in **C**, then $f^* \circ g^* = h$ in $\mathbf{C}^*$.

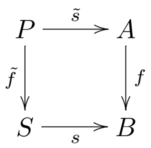
Exercise 4.0.41. *Find the dual notions of the following: mono, epi, isomorphism, initial, terminal.*

Exercise 4.0.42. *The dual of "pullback" is called a pushforward. Write a precise definition of pushforward.*

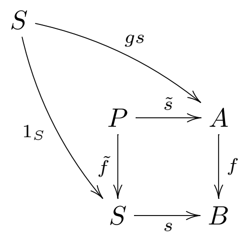
Let's prove some lemmas about pull-backs.

Lemma 4.0.43. *The pullback of an isomorphism is an isomorphism.*

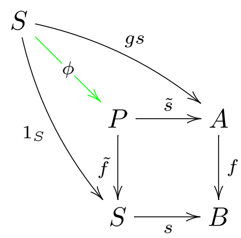
Proof. Suppose $f : A \to B$ is an isomorphism. Then there exists $g : B \to A$ so that $f \circ g = 1_B$ and $g \circ f = 1_A$. Let $s : S \to B$ be an arbitrary arrow and



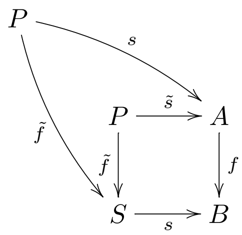
the pullback of f by s. Consider the commutative diagram



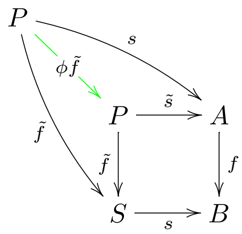
By definition we have a unique map ϕ so that



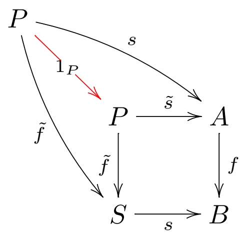
commutes. Thus, $\tilde{f}\phi = 1_S$. Now consider the commutative diagram



!!By tracing maps in the commutative diagram above we see that



commutes. But so too, obviously, does

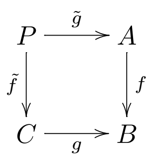


The definition of pullback says, however, that there can be only one map from P to P making that diagram commute. Thus, $\phi\tilde{f} = 1_P$. Since we have now that both $\tilde{f}\phi = 1_S$ and $\phi\tilde{f} = 1_P$, we have that $\tilde{f}$, the pullback of f by an arbitrary map is an isomorphism. Which is what we hoped to show. ☐

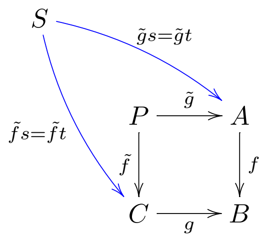
Exercise 4.0.44. *Prove the statement following the !! in the above proof*

Lemma 4.0.45. *The pullback of a mono by any map is again a mono.*

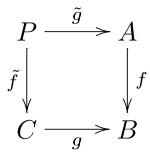
Proof. Suppose $f : A \to B$ is a mono and $g : C \to B$ is any map. Let



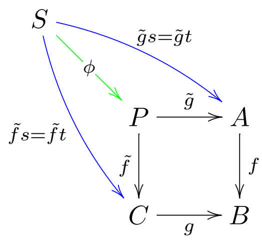
be the pullback diagram. Suppose that $\tilde{f}s = \tilde{f}t$. Then, $g\tilde{f}s = g\tilde{f}t$ which by the commutativity of the pullback diagram implies that $f\tilde{g}s = f\tilde{g}t$. Since f is monic, we thus have that $\tilde{g}s = \tilde{g}t$. In other words, the diagram



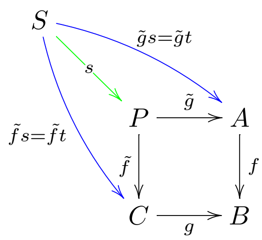
commutes. Since



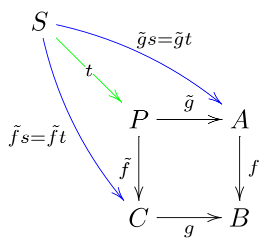
is a pullback, however, there exists a unique ϕ so that



commutes. Since both



and



commute however, we must have that $s = t = \phi$. Thus, since s and t were arbitrary, $\tilde{f}$ is monic. □

Exercise 4.0.46. *State and prove the duals of the preceding two lemmas.*
Earlier, we mentioned that there are categories in which epis are not onto as

functions of sets.

Exercise 4.0.47. *Give an example of a continuous function between two sets which is continuous and epi, but not onto (as a function of the underlying sets).*

4.1 Strong Monics and Epis

Clearly, if such examples exist as the exercise just above asks you to find, then there are categories in which an arrow might be both monic and epi, but, unlike in the category **SET**, is not an isomorphism. However, if we impose a slightly stronger requirement on mono, we do get a necessary and sufficient condition for an arrow to be an isomorphism.
An arrow $e : E \to B$ is said to be a strong mono if the pullback of it by any morphism $f : A \to B$ is not a non-isomorphic epi.

Theorem 4.1.1. *An arrow $f : A \to B$ is an isomorphism if and only if it is both a strong mono and an epi.*

Proof. Suppose f is an isomorphism. Then, we have seen that its pullback is an isomorphism. Thus, it is both a strong mono and an epi. On the other hand if f is both a strong mono and an epi, then its pullback by 1_B is again f, which means it must be an isomorphism (since no pullback can be a non-isomorphic epi, and it *is* epi) □

Exercise 4.1.2. *Prove the dual of this.*

Not all categories have pullbacks or pushforwards. The following definition for strong mono can be substituted for the above in case we are in such a category:

Lemma 4.1.3. *A mono $f : A \to B$ is strong if for every commutative diagram*

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

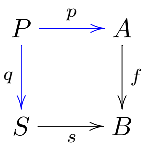
with r a monic arrow, there exists a unique map w so that

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\downarrow & \overset{w}{\nearrow} & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

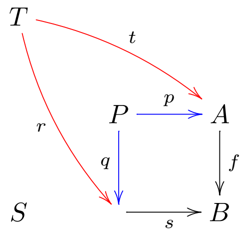
Proof. Suppose $f : A \to B$ is strong. Let s be an arbitrary arrow and r a monic arrow so that

$$\begin{array}{ccc} T & \xrightarrow{t} & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[s]{} & B \end{array}$$

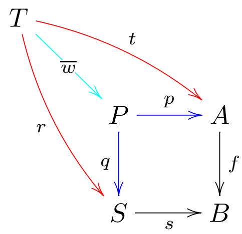
commutes. Let



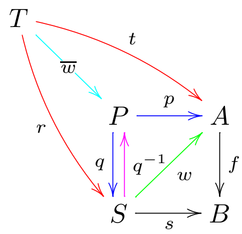
be the pullback of f by s. Then, the diagram



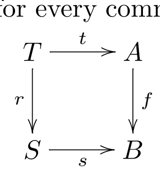
commutes.Thus, there exists a uniqe $\overline{w}$ so that



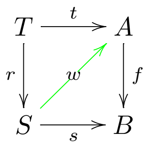
commutes. We have already proven that since r is epi, q must be as well.Denote by q^{-1} the inverse arrow of q. Set $w = p \circ q^{-1}$.



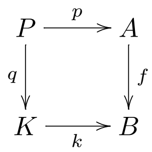
Now, $f \circ w = f \circ p \circ q^{-1} = s \circ q \circ q^{-1} = s$
$w \circ r = p \circ q^{-1} \circ r = p \circ \overline{w} = t$.
(Since $q \circ \overline{w} = r \Rightarrow q^{-1} \circ q \circ \overline{w} = q^{-1} \circ r \Rightarrow \overline{w} = q^{-1} \circ r$). Conversely, suppose that f is a monic arrow so that for every commutative diagram



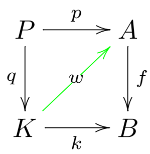
in which r is an epic arrow, there exists a unique map w so that in the diagram



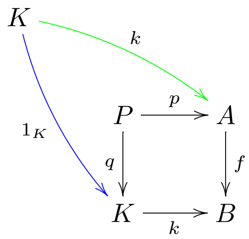
$f \circ w = s$ and $w \circ r = t$. Let k be any arrow and



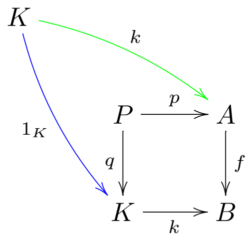
be the pullback of f by k. k is epic. Then, by hypothesis, there exists a unique w so that



$f \circ w = k$ and $w \circ q = p$. Then, the diagram



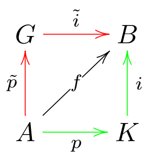
commutes. Thus, there exists a unique arrow ϕ so that



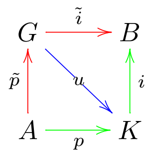
commutes. Since f is monic, so too is q. Since $q \circ \phi = 1_K$, q is an isomorphism. □

Lemma 4.1.4. *Suppose $f : A \to B$ and that $f = i \circ p$ where i is a strong mono and p is an epi. Suppose too that $f = \bar{i} \circ \bar{p}$. Then, $i = \phi\bar{i}$ and $p = \psi\bar{p}$ where ϕ and ψ are isomorphisms.*

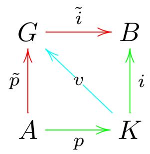
Proof. Consider, the following commutative diagram



Since p is a strong epi, and $\tilde{i}$ is monic, there exists an unique arrow u so that in the diagram



$u \circ \tilde{p} = p$ and $i \circ u = \tilde{i}$. Since we have assumed that $\tilde{p}$ is a strong epi, there exists a unique v so that in the diagram



$v \circ p = \tilde{p}$ and $\tilde{i} \circ v = i$. Now,

$$\begin{aligned}
i \circ u \circ v \circ p &= \tilde{i} \circ \tilde{p} && \tag{4.1}\\
&= i \circ p && \Rightarrow \tag{4.2}\\
u \circ v \circ p &= p && \text{since } i \text{ is monic} \tag{4.3}\\
u \circ v &= 1_K \circ p && \text{which implies} \tag{4.4}\\
u \circ v &= 1_K && \text{since } p \text{ is epic} \tag{4.5}
\end{aligned}$$

Similarly we can show that $v \circ u = 1_G$. Thus, both v and u are isomorphisms. In particular i and p are isomorphic to $\tilde{i}$ and $\tilde{p}$ respectively.

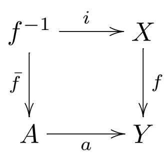
□

Exercise 4.1.5. *Show that in* **SET** *every mono is a strong mono and that every epi is a strong epi.*

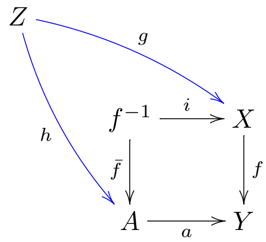
Pullbacks need not exist in a category. However, in the case of **SET** they do. Let's look at a special case.

Lemma 4.1.6. *Let $f : X \to Y$ be a function of sets and $A \subseteq Y$. If $a : A \to Y$ is the inclusion function, that is the function which takes each element of A to itself, then the pullback of f and a is the pre-image of A by f, $f^{-1}(A)$.*

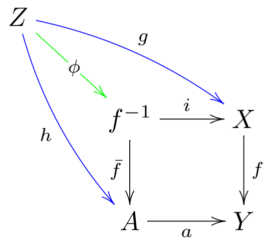
Proof. Let's set up the diagram first:



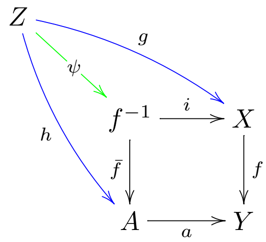
Here, i is the inclusion function, and $\bar{f}$ is the restriction of f to $f^{-1}(A)$. Now let Z be a set and g and h functions so that



commutes. That is, for every $x \in H$, $f(g(x)) = a(h(x))$. This allows us to say then that the diagram



commutes, where $\phi(x) := g(x)$. This definition makes sense because $f(g(x)) = h(x)$, and $\bar{f}\phi(x) = f(g(x))$. If ψ were another map which made

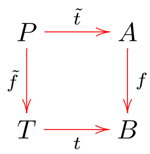


commute, then it would also follow that $i\phi = i\psi$ which implies that $\phi = \psi$ since i is one to one. □

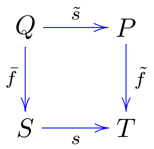
Definition 4.1.7. *In any category with pullbacks, if $\iota : H \to G$ is monic and $f : K \to G$ is an arrow, we will denote the pullback of f by ι $f^{-1}(H)$ and refer to it as pre-image of H by f.*

Pullbacks also have a nice transitive property which can be roughly stated as "the pullback of a pullback is a pullback." More precisely, we have:

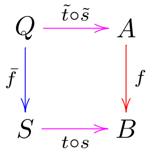
Lemma 4.1.8. *Let $f : A \to B$, $t : T \to B$ and $s : S \to T$ be arrows in a category. Let*



and

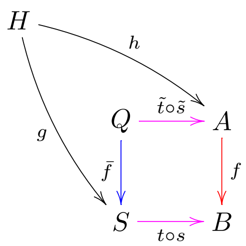


be pullback diagrams. Then,

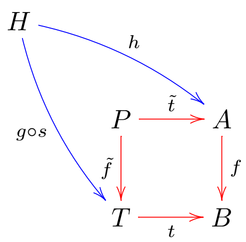


is a pullback diagram.

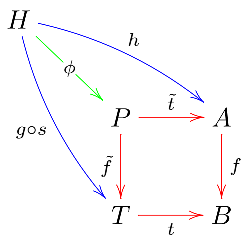
Proof. Suppose



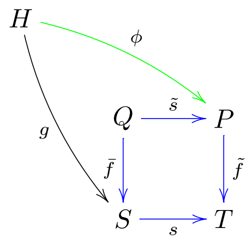
commutes. Then so too does



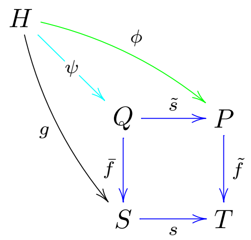
Thus, there exists a unique map ϕ so that



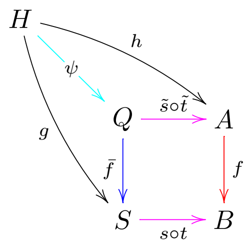
commutes. This in turn implies that



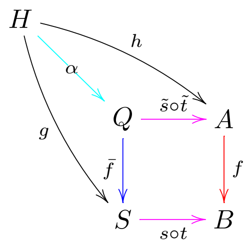
commutes. Thus, we have a unique map ψ so that



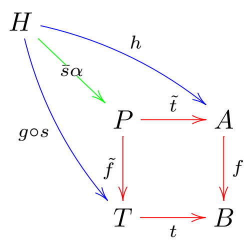
commutes. Thus, ψ is a map which makes



commute. What remains is to check that ψ is unique. So let us suppose that α is another arrow for which it is true that



commutes. Then we have that



commutes. But ϕ is the unique such map, whence $\bar{s}\alpha = \phi$. This implies, though, that $\bar{s}\psi = \bar{s}\alpha$, and $\bar{f}\alpha = g$ by assumption. But, ψ is the uniqe map which satisfies those last two equalities. Thus, $\alpha = \psi$, and so ψ is unique. □

Lemma 4.1.9. *The pullback of a strong mono is also a strong mono.*

Proof. Let $f : A \to B$ be a strong mono, and $t : T \to B$ be an arbitrary morphism. Let

$$\begin{array}{ccc} P & \xrightarrow{\tilde{t}} & A \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{t} & B \end{array}$$

be the pullback diagram. □

4.2 Equalizers

Another sort of universal diagram which plays a critical role in the categories we shall soon examine is the subject of the following definition:

Definition 4.2.1. *Let* $A \overset{f}{\underset{g}{\rightrightarrows}} B$ *be a pair of arrows. The equalizer of f and g is an object E and arrow e so that*

$$E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

commutes, and so that whenever

$$\begin{array}{ccc} A & \overset{f}{\underset{g}{\rightrightarrows}} & B \\ \uparrow{\scriptstyle t} & & \\ T & & \end{array}$$

commutes, there exists a unique ζ so that

$$\begin{array}{ccccc} E & \xrightarrow{e} & A & \overset{f}{\underset{g}{\rightrightarrows}} & B \\ & {\scriptstyle \zeta}\nwarrow & \uparrow{\scriptstyle t} & & \\ & & T & & \end{array}$$

commutes.

Definition 4.2.2. *A map e which is an equalizer of two arrow is sometimes called a regular arrow.*

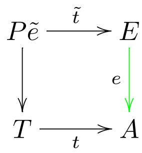
Let's prove an implication which will serve us later:

Lemma 4.2.3. *Every regular arrow is strong.*

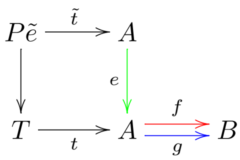
Proof. Let e be a regular arrow and

$$E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

be its equalizer diagram. Let $t : T \to A$ be an arbitrary arrow and



be a pullback diagram. Suppose that $\tilde{e}$ is epic. Consider now the diagram

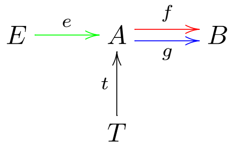


It commutes. This means

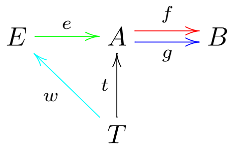
$$f \circ t \circ \tilde{e} = g \circ t \circ \tilde{e} \tag{4.6}$$

$$f \circ t = g \circ t \qquad \text{since } \tilde{e} \text{ is epic} \tag{4.7}$$

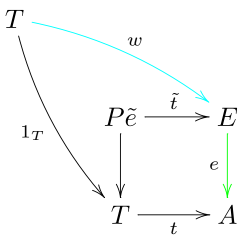
which implies that



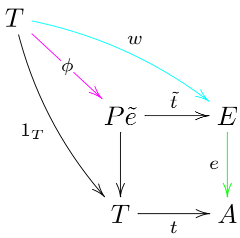
commutes. Thus, there exists a unique w so that



commutes. Thus, so too does



Thus, there exists a unique ϕ so that



commutes. Now,

$$\tilde{e} \circ \phi = 1_T \tag{4.8}$$
$$\tilde{e} \circ \phi \circ \tilde{e} = \tilde{e} = \tilde{e} 1_P \tag{4.9}$$
$$\phi \circ \tilde{e} = 1_P \qquad \text{since } \tilde{e}\text{, being the pullback of the monic } e \text{ is itself monic} \tag{4.10}$$

Thus, we have shown that if the pullback of e is epi it must also be an isomorphism. This is the definition of a strong arrow. □

Exercise 4.2.4. *Prove the statement in red in the preceding proof.*

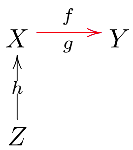
Exercise 4.2.5. *Show that if an arrow $f : A \to B$ is monic and there exists an arrow $g : B \to A$ so that $f \circ g = 1_B$, then f is an isomorphism.*

Exercise 4.2.6. *The dual of the equalizer of two maps is called the coequalizer of two maps. Write down its definition.*

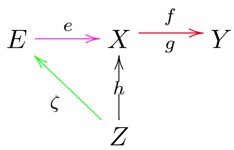
Let's prove some facts about equalizers, including a proof that the equalizer of any two maps in **SET** exists.

Lemma 4.2.7. *Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be two functions. Then the equalizer of f and g is the set $E := \{x \in X | f(x) = g(x)\}$ and the function $e : E \to X$ defined by $e(w) = w$ - that is - the inclusion.*

Proof. First, Check that $fe = ge$. Now, let $h : Z \to X$ be a function such that



commutes. In this case, this is just a different way of saying that $f(h(z)) = g(h(z))$ for every $z \in Z$. This is important, though, because it insures that

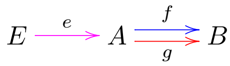


commutes where ζ is defined by the assignment $z \mapsto h(z)$. The proof is completed by working the following exercises and the more general lemma which follows them. □

Exercise 4.2.8. *Prove the first statement in red in the proof above.*

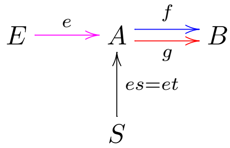
Exercise 4.2.9. *Prove the second statement in red in the proof above. Include in your argument that the definition of ζ makes sense. (What is the issue here?)*

Lemma 4.2.10. *Let*

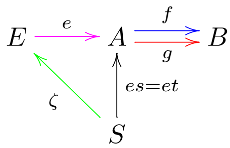


be an equalizer diagram. Then e is monic.

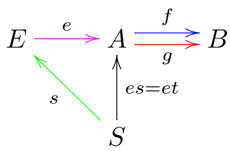
Proof. Suppose $e \circ s = e \circ t$ for a pair of arrows s and t from S to E. Then, of course, $f \circ e \circ s = g \circ e \circ s = g \circ e \circ t = f \circ e \circ t$. In other words



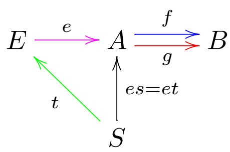
commutes. Since E is an equalizer, there exists a unique map ζ such that



commutes. But, both



and



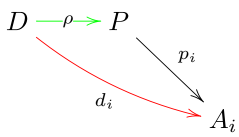
commute. Thus $\zeta = s = t$. Thus, e is monic. □

Exercise 4.2.11. *Put the last two lemmas together to complete the proof that the construction outlined in the first of those two is really an equalizer in* **SET**.

4.3 Products

A third sort of universal diagram is known as the product.

Definition 4.3.1. *Let $\{A_i\}_{i \in I}$ be a collection of objects. The product of these objects, if it exists, is defined to be an object P and arrows $\{p_i : P \to A_i\}_{i \in I}$ with the following property: For any object D and arrows $\{d_i : D \to A_i\}$ there is a unique map ρ so that for every $i \in I$*

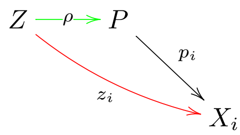


commutes.

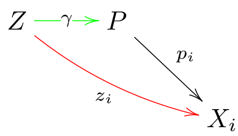
In **SET**, as in many, many categories, the product is a relatively familiar construction: It is the set of all I-tuples of elements drawn from the sets. More precisely, we have

Lemma 4.3.2. *Let $\{X_i\}_{i\in I}$ be a collection of sets. Then, the product is given by $P := \{(x_i)_{i\in I} | x_i \in X_i\}$ and arrows $p_j(x_i)_{i\in I} \mapsto x_j$.*

Proof. Let $\{X_i\}$ be a collection of sets. Let Z be a set and $z_i : Z \to X_i$ a collection of maps, one for each $i \in I$. Define $\rho : Z \to P$ by $z \mapsto (z_i(z))_{i\in I}$. Then, clearly



commutes for each $i \in I$. Now, suppose that γ is a function so that



commutes. Then, for each $i \in I$, $p_i(\gamma(z)) = z_i(z)$. That is, the i^{th} component of $\gamma(z) = z_i(z)$. This is just the definition of ρ though. Thus, ρ is unique. □

Exercise 4.3.3. *Justify the statement highlighted in red in the proof above.*

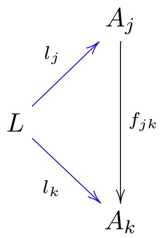
Exercise 4.3.4. *The dual construction in this case is generally known as the coproduct. Give its definition.*

Exercise 4.3.5. *Show that in* **SET***, the coproduct of $\{X_i\}_{i\in I}$ is the disjoint union of the X_i. (Recall that the disjoint union of two sets X and Y is the set of all elements in X and all elements in Y with any elements of $X \bigcap Y$ listed twice - so that if $X := \{1, 2\}$ and $Y := \{1, 2, 3\}$ the disjoint union $X \coprod Y = \{1, 1, 2, 2, 3\}$).*

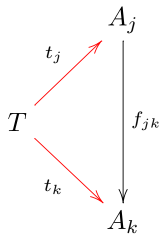
4.4 Limits

Equalizers, products and pullbacks are in fact particular examples of a more general -and complicated - universal diagram: the limit diagram.

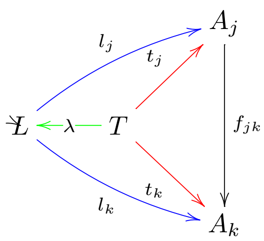
Definition 4.4.1. *Let $\{A_i\}_{i\in I}$ be a collection of objects and $\{f_{jk} : A_j \to A_k\}_{j,k\in I}$ a collection of arrows between them. Then, the limit of these collections, if it exists, is an object L and arrows $\{l_i\}_{i\in I}$, one for each $i \in I$ so that for every f_{jk} the diagram*



commutes, and, given any other object T *and arrows* $\{t_i\}_{i \in I}$ *so that*



commutes for every j, k*, there exists a unique map* λ *so that*



commutes.

Exercise 4.4.2. *Justify the statement: "Equalizers, products, and pullbacks are particular examples of limits." Explain in each case what objects and arrows are playing what role in the definition of limit.*

Exercise 4.4.3. *The dual notion to that of limit is usually refferred to as colimit. Write down the definition of colimit.*

Exercise 4.4.4. *Can the terminal or initial object of a category be regarded as a limit or colimit? How?*

Chapter 5

The Category of Groups

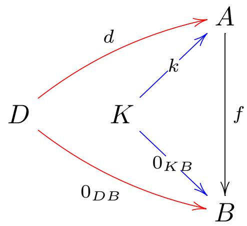
5.1 Introduction

We now come to the first of the categories which, together, are the focus of this text. Staying faithful to the philosophy outlined in the introduction, we will first define the category in terms of its properties, and only then will we examine the "internal" properties of the objects in the category. The category of groups contains as a sub-category an extremely important collection of objects and arrows known as "Abelian groups" in honor of the great Norwegian mathematician Abel. The first property which distinguishes **AbG** from many other categories is that it possesses a "zero" object, which we shall denote - perhaps not very creatively - 0. It is both initial and terminal in the category, and not surprisingly plays a vital roll. Let A and B be objects in **AbG**. Then there exists a unique map from A to 0 and a unique map from 0 to B. Let us from now on denote the composition of these two as 0_{AB}. With this notation we can define a particular, and important sort of equalizer - and dually coequalizer.

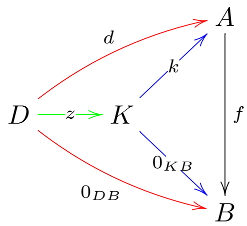
5.2 Definition

Definition 5.2.1. *Let A and B be objects in a category with a zero object. Let $f : A \to B$ be an arrow in such a category. Then, the kernel of f is the equalizer of f and 0_{AB}.*

Note that because $f \circ 0 = 0$, we can express the fact that $k : K \to A$ is the kernel of f in the following way:
Given any commutative diagram



there exists a uniqe arrow z so that



commutes.

Exercise 5.2.2. *Define the dual notion to kernel, the cokernel.*

Exercise 5.2.3. *Show that the unique map from* 0 *to* A *is always monic, and that the unique map from* B *to* 0 *is always epic. What can you say about the unique map* 0_{AB}*?*

Exercise 5.2.4. *A strong epi is the dual notion to that of strong mono defined in the previous chapter. Prove that the unique map to the zero object is always a strong epi.*

Viewed from the proper perspective, the category of Abelian groups can be largely understood by understanding the nature of its Hom sets.
In the category of Abelian Groups, which we will denote **AbG** all Hom sets are equipped with a structure which makes them a bit like the integers:

Definition 5.2.5. *Let* A *and* B *and* C *be objects in* **AbG**. *Then,* $Hom(A, B)$ *is equipped with an assignment* $+ : Hom(A, B) \times Hom(A, B) \to Hom(A, B)$ *which satisfies the following for every* $f, g \in Hom(A, B)$ *and* $h \in Hom(B, C)$*:*

1. $f + g = g + f$
2. $f + (g + h) = (f + g) + h$
3. $f + 0_{AB} = f$
4. $h \circ (f + g) = h \circ f + h \circ g$
5. *There exists an element* $\bar{f} \in Hom(A, B)$ *so that* $f + \bar{f} = 0_{AB}$.

Exercise 5.2.6. *Show that the element* $\bar{f}$ *defined just above is uniqe.*

Definition 5.2.7. *Any category whose* Hom *sets meet the criteria listed in the last definition is referred to as preadditive.*

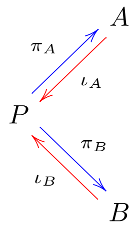
In preadditive categories, finite products and coproducts have an unusual relationship:

Lemma 5.2.8. *Let A and B be objects in* $\mathbf{C}$*, a preadditive category. Then, the following are equivalent:*

1. *The product $A \prod B$ exists*

2. *The coproduct $A \coprod B$ exists*

3. *There exists an object P and arrows*



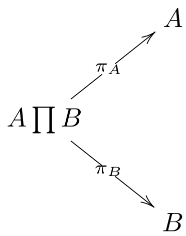
so that

(a) $\pi_A \circ \iota_A = 1_A$ *and* $\pi_B \circ \iota_B = 1_B$

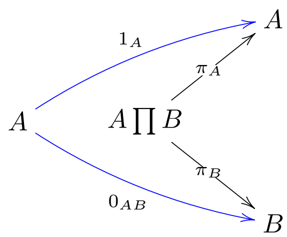
(b) $\pi_A \circ \iota_B = 0_{BA}$ *and* $\pi_B \circ \iota_A = 0_{AB}$

(c) $\iota_A \circ \pi_A + \iota_B \circ \pi_B = 1_P$.

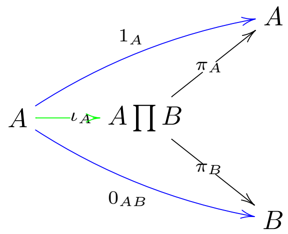
Proof. Suppose (1). Let



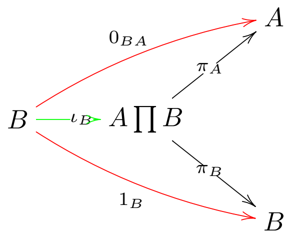
be the product diagram. Consider the diagram



Then there exists a unique map ι_A so that



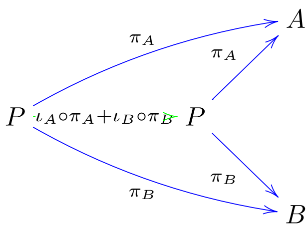
commutes. Similarly we have a unique map ι_B so that



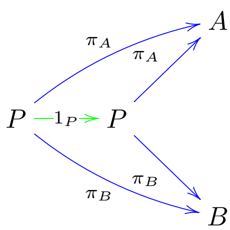
commutes. Now, a short calculation shows that

$$\pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_A + 0_{PA} = \pi_A \quad \text{and,} \tag{5.1}$$
$$\pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_B + 0_{PB} = \pi_B \tag{5.2}$$

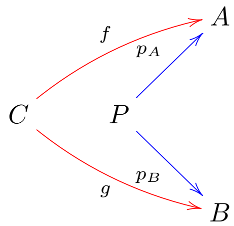
In other words,



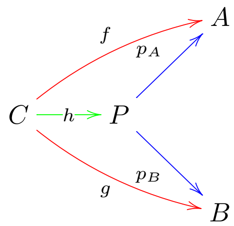
commutes. But, clearly, 1_P is the unique map which makes



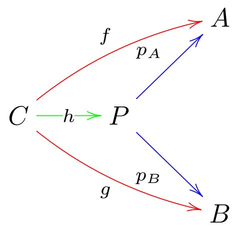
commute. Thus, $\iota_A \circ \pi_A + \iota_B \circ \pi_B = 1_P$. Now suppose (3). Suppose C is an object and f and g arrows and consider the diagram:



For ease of notation, let us set $h = \iota_A \circ f + \iota_B \circ g$. Then, we have that



commutes. Suppose now that h' is a map so that



commutes. Then,

$$h' = 1_P \circ h' = (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \circ h' \tag{5.3}$$
$$= \iota_A \circ \pi_A \circ h' + d\iota_B \circ \pi_B \circ h' \tag{5.4}$$
$$= \iota_A \circ f + \iota_B \circ g \tag{5.5}$$
$$= h \tag{5.6}$$

In other words, h is unique. Since C, f and g were arbitrary, P and p_A and p_B constitute a product of A and B. □

Exercise 5.2.9. *In the proof above we proved* (1) ⇔ (3). *Show, using duality,* (2) ⇔ (3).

Exercise 5.2.10. *Prove the statement highlighted in red in the proof.*

Exercise 5.2.11. *Show that*

1. $\iota_A = ker\pi_B$
2. $\iota_B = ker\pi_A$
3. $\pi_A = coker\iota_B$
4. $\pi_B = coker\iota_A$

Definition 5.2.12. *A product which satisfies (3) in the statement of the above lemma is called a biproduct.*

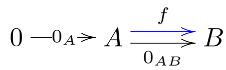
Definition 5.2.13. *A category* **C** *is said to be Abelian if it is additive and, the following hold:*

1. *Every two objects A and B of* **C** *have a biproduct*
2. *Every monomorphism is a kernel of some map, and every epimorphism is the cokernel of some map*
3. *Every map has both a kernel and cokernel*

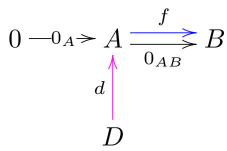
Lemma 5.2.14. *In an Abelian category, the following are equivalent for an arrow $f : A \to B$:*

1. *f is monic*
2. *$Ker f = 0$*
3. *For all $g : C \to A$, if $f \circ g = 0$, then $g = 0$.*

Proof. (1) $\Rightarrow$ (2): Consider the diagram



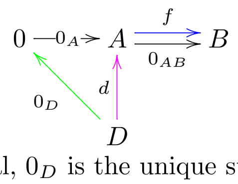
It is commutative. Let $d : D \to A$ be a map so that



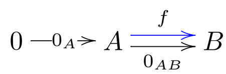
commutes. Then,

$$
\begin{aligned}
f \circ d &= 0_{AB} \circ d && (5.7)\\
&= 0_{DB} && (5.8)\\
&= f \circ 0_{DA} && (5.9)
\end{aligned}
$$

which implies that $d = 0_{DA}$ since f is monic. Thus,

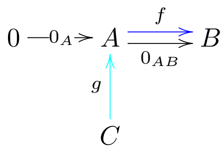


commutes. Since 0 is terminal, 0_D is the unique such map. Thus,

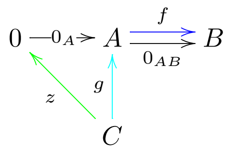


is an equalizer diagram.
(2) $\Rightarrow$ (3): Suppose $g : C \to A$ is an arrow and that $f \circ g = 0$. Then, we have

the commutative diagram



where the top line is the equalizer of f and 0_{AB}. Thus, there exists a uniqe map z so that



commutes. As before, however, since the target of z is 0, z must equal 0_C. But this in turn implies that $g = 0_A \circ 0_C = 0_{CA}$.
(3) ⇒ (1): Suppose (3). Suppose g and h are arrows from C to A so that

$$f \circ g = f \circ h \qquad \text{then,} \tag{5.10}$$

$$f \circ g - f \circ h = 0_{CD} \qquad \text{whence,} \tag{5.11}$$

$$f \circ (g - h) = 0 \qquad \text{so, by assumption} \tag{5.12}$$

$$g - h = 0_{CD} \qquad \text{and thus} \tag{5.13}$$

$$g = h \qquad \text{as required} \tag{5.14}$$

□

Exercise 5.2.15. *Show, dually to the last lemma, that the following are equivalent for an arrow $f : B \to A$:*

1. *f is an epi*
2. *$coker f = 0$*
3. *For every $g : A \to C$, if $g \circ f = 0$, then $g = 0$.*

5.3 The Isomorphism Theorems

Lemma 5.3.1. *Every arrow $f : A \to B$ in* **AbG** *can be uniquely factored as the composition of a strong epic arrow p followed by a monic arrow i.*

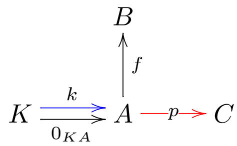
Proof. Let

$$K \xrightarrow{k} A \underset{0_{AB}}{\overset{f}{\rightrightarrows}} B$$

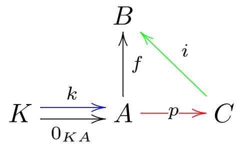
be the kernel of f. Note that $f \circ k = 0_B$. Let

$$K \underset{0_{KA}}{\overset{k}{\rightrightarrows}} A \xrightarrow{p} C$$

be the cokernel of k. Then



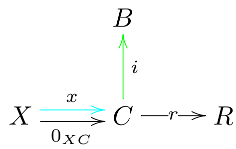
commutes, which implies that there exists a unique map i so that



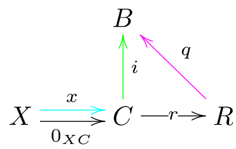
commutes. In other words, that $f = i \circ p$. Since p is a cokernel, it is a strong epic. Thus, we have left to prove only that i is monic. So, suppose that $x : X \to C$ is a map so that $i \circ x = 0$. Let

$$X \underset{0_{XC}}{\overset{x}{\rightrightarrows}} C \xrightarrow{r} R$$

be the cokernel diagram for x. Then, by assumption,



commutes. Thus, there exists a unique map q so that



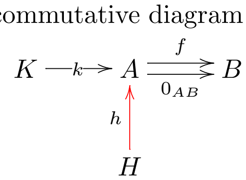
commutes. Now, since both r and p are epic, so is $r \circ p$. Since we are in an Abelian category, there must exist an arrow $h : H \to A$ so that

$$H \underset{0_{HR}}{\overset{h}{\rightrightarrows}} A \xrightarrow{r \circ p} R$$

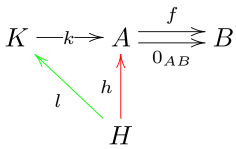
is a cokernel diagram. Now,

$$\begin{aligned} f \circ h &= i \circ p \circ h && && (5.15) \\ &= q \circ r \circ p \circ h && \text{since i} = q \circ r && (5.16) \\ &= q \circ 0_{HR} && r \circ p \circ h = 0 && (5.17) \\ &= 0 && \text{the composition of} && (5.18) \end{aligned}$$

In other words we have the commutative diagram



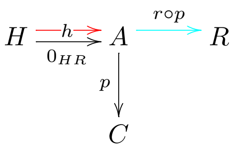
which, of course, implies that there exists a unique map l so that



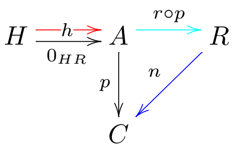
commutes. Now,

$$p \circ h = p \circ k \circ l \tag{5.19}$$
$$= 0_{KC} \circ l \tag{5.20}$$
$$= 0_{HC} \tag{5.21}$$

In other words,



commutes. This implies that there exists a unique n so that



commutes. In other words, that $n \circ r \circ p = p = 1_A p$. But remember, that p, being the cokernel of k is epi. Thus, $n \circ r = 1_A$, which implies that r is monic. But, r, by defintion, is a map so that $r \circ x = 0_X$. Since r *is* monic, we must have that $x = 0$. At the start of all this, though, we let x be an arbitrary map with the property that $i \circ x = 0$. Since this implies that x must also be 0 we see that i is monic. The uniqueness of this factoriztion is an immeadiate consequence of 4.1.4 □

Exercise 5.3.2. *Prove all statements highlighted in red in the above proof.*

Exercise 5.3.3. *Show that in* **SET** *there is a similar factorization for every function.*

In the above proof, the object C and map i were the cokernel of the kernel of f. This object and map, and their dual will be of interest to us, and so merit particular names.

Definition 5.3.4. *Let $f : A \to B$ be an arrow in an Abelian category. Then $C := ker(coker(f))$ and its map $j : C \to B$ are known as the image of the map f. We will denote it $im(f)$. Dually, the object $D := coker(ker(f))$ and the map $p : A \to D$ are known as the coimage of f. We will denote this object $coim(f)$.*

Exercise 5.3.5. *Prove the dual of 5.3.1.*

If you work this exercise you will discover that in this case $f = \bar{i}\bar{p}$ where $\bar{p}$ is the coimage of f. Something is true about these dual objects - image and coimage - which is virtually never true: they are isomorphic.

Exercise 5.3.6. *We have defined the kernel of the cokernel to be the image. What can you say about the kernel of the kernel of an arrow in an Abelian category?*

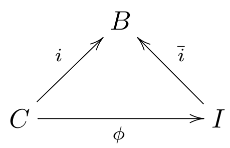
Exercise 5.3.7. *What can you say about the cokernel of the cokernel of an arrow in Abelian category?*

Exercise 5.3.8.

Exercise 5.3.9. *Show that the dual of an epi monic factorization $i \circ p$, as above, is again an epi monic factorization.*

Lemma 5.3.10. *Let $f : A \to B$ be an arrow in an Abelian category. Then, $im(f) \simeq coim(f)$.*

Proof. By 5.3.5, and 5.3.1, $f = i \circ p = \bar{i} \circ \bar{p}$, where i is the image of f and $\bar{i}$ is the coimage. Since i and p (respetively $\bar{p}$) is a kernel (respectively cokernel), i (respectively p) is a strong mono (respectively epi) by 4.2.3 and its dual. **??** implies that $i : C \to B$ and $\bar{i} : I \to B$ are isomorphic. That is, there exists an isomorphism ϕ so that the diagram



commutes. In other words, the coimage C and image I are isomorphic. ☐

Exercise 5.3.11. *Let $f : A \to 0$. Show that the kernel of f is isomorphic to A.*

Exercise 5.3.12. *Let $g : 0 \to A$. Show that the cokernel of g is isomorphic to A.*

Lemma 5.3.13. *Suppose $f : A \to B$ is an arrow in an Abelian category.*

1. *If f is monic, then the image of f is isomorphic to A.*
2. *If f is epic, then the image of f is isomorphic to B.*

Proof. This is left as a pair of exercises. ☐

Exercise 5.3.14. *Prove statement 1 in 5.3.13.*

Exercise 5.3.15. *Prove statement 2 in 5.3.13.*

Recall that an equivalence relation on a set S is a collection of subsets of $S \times S$, which for notational ease we will refer to as $\sim$, i.e. $\sim \subseteq S \times S$, so that the following hold:

1. $< x, x > \in \sim$ for every $x \in S$
2. If $< x, y > \in \sim$, then $< y, x > \in \sim$

3. If $< x, y > \in \sim$ and $< y, z > \in \sim$, then $< x, z > \in \sim$.

These are known as the reflexive, symmetric and transitive properties respectively. When such a subset $\sim$of $S \times S$ is specified we will often refer to an element $< x, y > \in \sim$ using the notation $x \sim y$.

Exercise 5.3.16. *Recall that a partition of a set X is a collection of subsets $\{W_\alpha\}_{\alpha \in A}$ so that*

$$\bigcup_{\alpha \in A} W_\alpha = X$$

and

$$W_\alpha \bigcap W_{\alpha'} = \emptyset \forall \alpha \neq \alpha' \in A.$$

Show that every equivalence relation determines a partition and that every partition determines an equivalence relation.

Lemma 5.3.17. *Let X be an object in a category. Let $j : J \to X$ and $i : I \to X$ be two monics. Let us say that $i \sim j \Rightarrow i \simeq j$. Then, $\sim$ is an equivalence relation.*

We check each of the properties in turn:

1. Reflexivity: $i \sim i$ since $i = 1_I i$ and 1_I is an isomorphism.

2. Symmetry: Suppose $i \sim j$. Then there exists an isomorphism ϕ so that $i = \phi \circ j$. But then $\phi^{-1} \circ i = j$, and ϕ^{-1} is an isomorphism.

3. Transitivity: Suppose $i \sim j$ and $j \sim k$. Then there exists isomorphisms ϕ and ψ so that $i = \phi \circ j$ and $j = \psi \circ k$. This implies though that $i = \phi \circ \psi \circ k$ and since the composition of ϕ and ψ, being the composition of two isomorphisms, is, itself, an isomorphism, we have $i \sim k$.

Definition 5.3.18. *An equivalence class of monics with target X, as defined above, is called a subobject of X.*

There is a subtle distinction to be made here. If we refer to only a subobject H of G, then we mean that H is in fact an equivalence class, as above. We may also, however, refer to an object and arrow $h : H \to G$ as a subobject. In this case we do not mean the equivalence class to which H belongs, but instead just the object H with the specific monic h which defines how H sits in G. Let's look at an example in the category **SET**. There is a sense in which *every* two element set is a subset (subobject) of every three element set. That is, the set $\{a, b\}$ is contained in the set $\{d, e, f\}$ if we identify a with d and b with e. It can also be considered a subset if we identify a with e and b with d. These "identifications" define functions $a \mapsto d$ etc. and thus, places $\{a, b\}$ in the same equivalence class as $\{d, e\}$. Each viewpoint has its advantages.

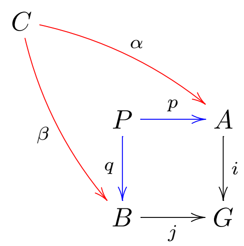
Lemma 5.3.19. *The pullback P of two monics $i : A \to G$ and $j : B \to G$ has the following properies:*

1. *It is a subobject of both A and B.*

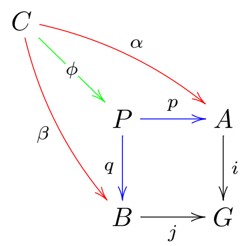
2. *If $c : C \to G$ is a subobject of both A and B , then C is a subobject of P.*

Proof. 1. This follows from **??**

2. Suppose C is a subobject of both A and B. Consider the diagram



It commutes. Since P is a pullback there exists a unique ϕ so that



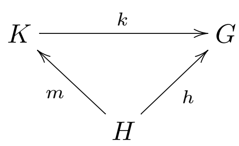
commutes. Since α is monic so too must ϕ be monic.

∎

In this context, where we are discussing the pullback of two monics, the pullback P, $\tilde{i}$, $\tilde{j}$ will often be written $A \bigcap B$.
It can be shown that the collection of subobjects of an object forms a lattice:

Exercise 5.3.20. *Show that the collection of equivalence classes of subobjects forms a lattice.*

Exercise 5.3.21. *Let G be an object in* **AbG**. *Let $\mathcal{S} := \{< H, h > | h : H \to G \text{is a monic}\}$. Definie $< H, h > \leq < K, k >$ if and only if there exists a monic arrow m so that*

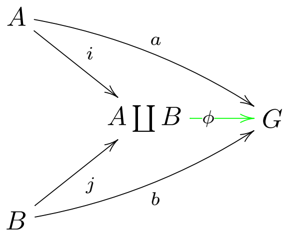


commutes. Show that the pair $< \mathcal{S}, \leq >$ constitute a lattice.

Exercise 5.3.22. *What is the relationship if any between the two lattices defined above?*

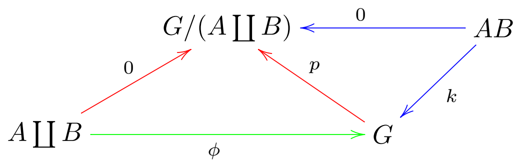
As 5.3.19 shows, the greatest lower bound is the isomorphism class of the pullback of any two representatives of the subobjects. Constructing the least upper bound of two subobjects is a little more complicated. It is doable, however, at least if you're in an Abelian category. This time we will work in the partially ordered set defined in 5.3.21.

Lemma 5.3.23. *Let $a : A \to G$ and $b : B \to G$ be two monics. Then in the partially ordered set of monics with target G, the least upper bound of A and B is the image AB of the unique map ϕ which makes the diagram*

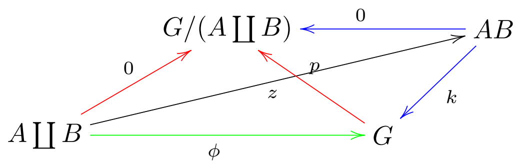


commute.

Proof. We first show that $A \leq AB$. Consider the image diagram for ϕ



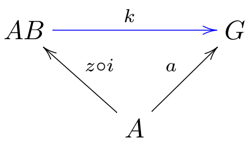
where the red diagram is the cokernel of ϕ and the blue the kernel of p - thus the image of ϕ. Since $p \circ \phi = 0$, there exists a unique arrow z so that



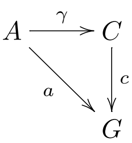
commutes. That is, z is the uniqe arrow so that $k \circ z \ = \phi$. Thus,

$$k \circ z \circ i = \phi \circ i = a$$

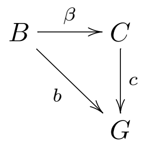
which implies that $z \circ i$ is monic since a is. The argument is completed by noting that



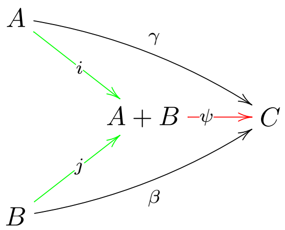
commutes. The proof that $B \leq AB$ is similar.
Now, suppose that C is an upper bound for the set $\{< A, a >; < B, b >\}$. Then we have the commutative diagrams



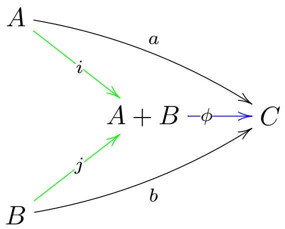
and,



where β and γ are monic. This implies that there exists a unique ψ so that



commutes; where the green diagram is the sum diagram for A and B. Let's reset the table, so to speak: Let



be the sum diagram (in green again) and ϕ the unique arrow making the diagram commute. Now,

$$\psi \circ i = \gamma \qquad \text{(thus,)} \tag{5.22}$$
$$c \circ \psi \circ i = c \circ \gamma \qquad \text{(thus,)} \tag{5.23}$$
$$= a \tag{5.24}$$

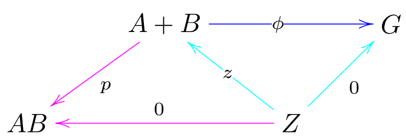
Similarly,

$$\psi \circ j = \beta \qquad \text{(thus,)} \tag{5.25}$$
$$c \circ \psi \circ j = c \circ \beta \qquad \text{(thus,)} \tag{5.26}$$
$$= b \tag{5.27}$$

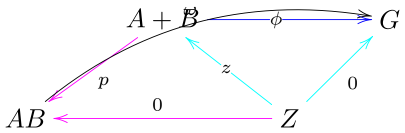
However, ϕ is the unique arrow so that

$$\phi \circ i = a \qquad \text{(and)} \tag{5.28}$$
$$\phi \circ j = b \tag{5.29}$$

Thus, it must be that $c\psi = \phi$. Now, let



be the coimage diagram for ϕ. Then, as we have seen, there exists a unique monic arrow ω so that



Now, we can see that

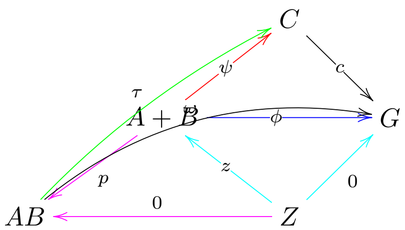
$$\phi \circ z = 0 \quad \text{(which implies that)} \tag{5.30}$$

$$c \circ \psi z = 0 \quad \text{(since } c \circ \psi = \phi \text{)} \tag{5.31}$$

$$\psi \circ z = 0 \quad \text{(since } c \text{ is monic)} \tag{5.32}$$

$$\tag{5.33}$$

Thus, there exists a unique arrow τ so that

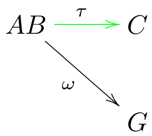


commutes - that is, so that $\tau \circ p = \psi$. (Why?) Now,

$$c \circ \tau \circ p = \omega \circ p \quad \text{(which implies that)} \tag{5.34}$$

$$c \circ \tau = \omega \quad \text{(since } p\text{, being a cokernel, is epi)} \tag{5.35}$$

Thus, τ is monic **??**, and



commutes. In other words, $AB \leq C$. Since C was an arbitrary upper bound for $\{< A, a >; < B, b >\}$ we have proven that AB is the least upper bound.

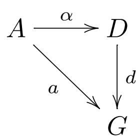
□

Exercise 5.3.24. *Prove the statement in red in the lemma above.*

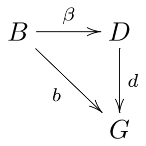
Exercise 5.3.25. *Prove the statement highlighted in red in the proof of the preceding lemma.*

We complete our assertion that the object which we denoted AB above is the least upper bound of the two subobjects of G, A and B.

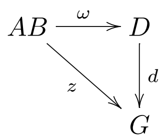
Lemma 5.3.26. *Let*



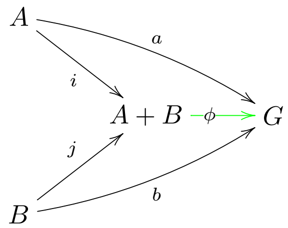
and



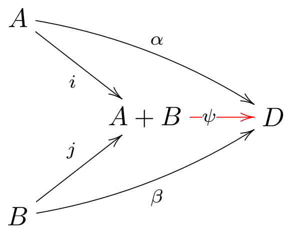
be commutative diagrams of monics. Let AB be as above. Then, there exists a commutative diagram of monics



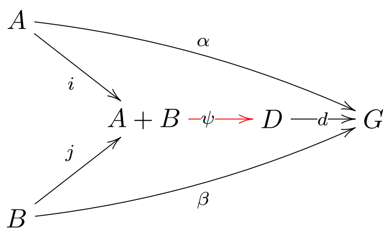
Proof. Let ϕ be the unique arrow making



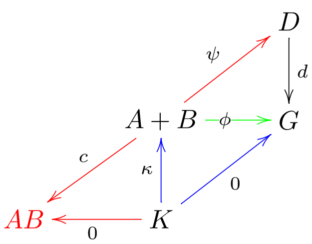
commute. Let ψ be the unique arrow making



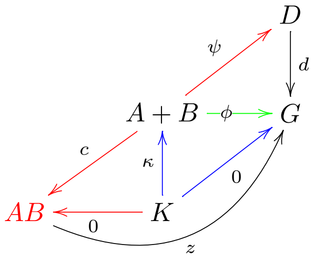
commute. Then,



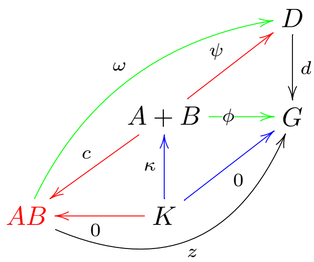
also commutes. Consider the diagram



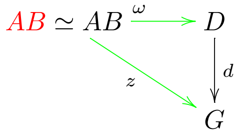
where the blue diagram is the the kernel of ϕ and the red is the cokernel of the kernel of ϕ, that is, the co-image. I named the co-image AB since, as we proved in **??** it is isomorphic to the image, but I have colored it red so as to note that it is not *really* AB. First, as we have seen before, since $\phi \circ \kappa = 0$ and c is the cokernel of κ there exists a uniqe map z so that $z \circ c = \phi$. That is



commutes. Now, since $d \circ \psi = \phi$, we have that $d \circ \psi \circ \kappa = 0$. But d is monic. Thus, $\psi\kappa = 0$. Again, this implies there exists a unique arrow, let's call it ω so that, in this case, $\omega \circ c = \psi$. In other words,



commutes. Note now that since $d \circ \omega \circ c = \phi$ and that z is the unique arrow so that $z \circ c = \phi$ it must be that $d \circ \omega = z$. But, a careful reading of the proof of **??** reveals that z is monic. Thus, so too is ω. All together, than we have that



commutes. This is, finally, what we claimed to be true. □

Exercise 5.3.27. *Prove all statements written in red in the proof of the preceding lemma.*

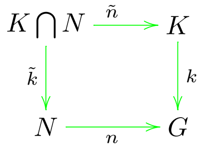
Definition 5.3.28. *The dual notion to that of subobject is referred to as quotient object. That is, if $p : G \to H$ is epi we will say that H is a quotient object of G.*

5.4 The Second Isomorphism Theorem

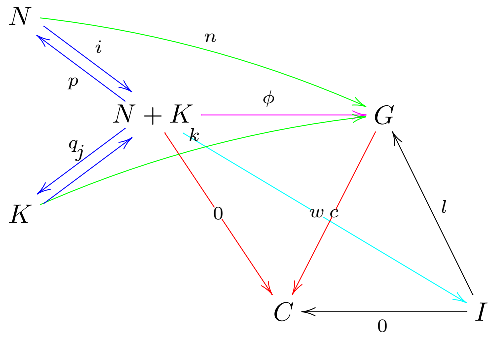
In this section and thereafter, we will have cause to use the following notation: If $f : G \to H$, then we will write the object part of the cokernel of f as H/G. As well, for the sake of simplicity, we will regard the subobject of an object D to be a pair $< M, m >$ where $m : M \to D$ is a monic.

Theorem 5.4.1. *Let G be an object in an Abelian category. Let $n : N \to G$ and $k : K \to G$ be subobjects of G. Then there is an isomorphism of objects $K/N \cap K \simeq NK/N$.*

Proof. We first must make precise what we mean by $K/N \cap K$ and NK/N. Let's tackle $K/N \cap K$ first. Consider the pullback diagram:



By $K/N \bigcap K$ we shall mean the cokernel of $\tilde{n}$. The case of NK/N is a bit more complicated. Consider this large diagram:



where the blue diagram is the biproduct diagram of N and K, ϕ is the unique arrow guaranteed by the definition of product and the existence of the arrows n and k. The red diagram is the cokernel diagram of ϕ, l is the kernel of c , and w the arrow guaranteed to exist by the universal property of the kernel. Note that we have the following equalities:

1. $\phi \circ i = n$
2. $\phi \circ j = k$
3. $l \circ w = \phi$

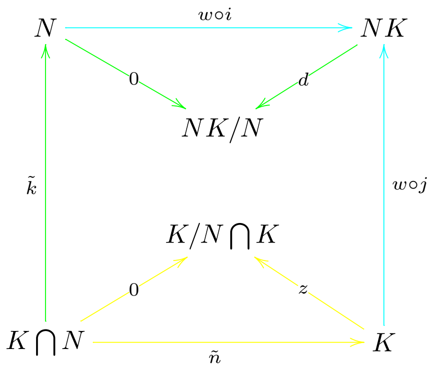
By NK/N, then we shall mean the object part of the cokernel of $w \circ i$. Now, since $l \circ w = \phi$, we have

$$l \circ w \circ j \circ \tilde{n} = \phi \circ j\tilde{n} \tag{5.36}$$
$$= k \circ \tilde{n} \qquad \text{by (2)} \tag{5.37}$$
$$= n \circ \tilde{k} \qquad \text{by commutativity of the pullback} \tag{5.38}$$
$$= \phi \circ i \circ \tilde{k} \qquad \text{by definition of } \phi \tag{5.39}$$
$$= l \circ w \circ i \circ \tilde{k} \qquad \text{by (3)} \tag{5.40}$$

Since l is monic (remember it is the kernel of c) we may conclude that

$$w \circ j \circ \tilde{n} = w \circ i \circ \tilde{k}.$$

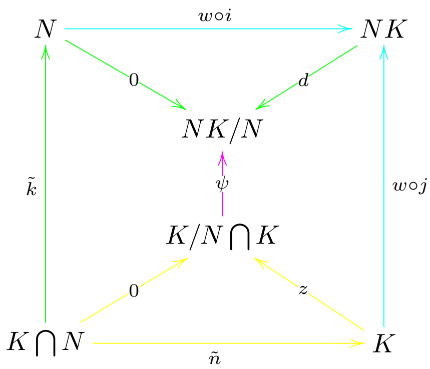
In other words,



where the green and yellow diagrams are the cokernel diagrams, is commutative. Now,

$$d \circ w \circ j \circ \tilde{n} = d \circ w \circ i \circ \tilde{k} \tag{5.41}$$
$$= 0 \circ \tilde{k} \tag{5.42}$$
$$= 0 \tag{5.43}$$

thus there exists a unique arrow ψ, so that



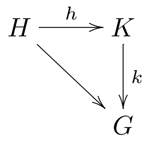
commutes. Now,

$$\psi \circ z = d \circ w \circ j \quad \text{which implies} \tag{5.44}$$
$$\psi \circ z \circ q = d \circ w \circ j \circ q \tag{5.45}$$
$$\psi \circ z \circ q = d \circ w \circ (1_{N+K} - i \circ p) \quad \text{since } N + K \text{ is a biproduct} \tag{5.46}$$
$$\psi \circ z \circ q = d \circ w \circ -d \circ w \circ i \circ p \tag{5.47}$$
$$\psi \circ z \circ q = d \circ w \quad \text{since } d \circ w \circ i = 0 \tag{5.48}$$
$$\tag{5.49}$$

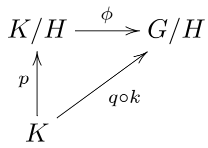
But recall (lemma 5.3.10, exercise5.3) that w is an epimorphism. Thus, since d is a cokernel it too is an epi, so, the composition of epis being an epi, $d \circ w : K \to NK/N$ is an epi. By definition, the kernel of this arrow □

5.5 The Third Isomorphism Theorem

Lemma 5.5.1. *Suppose that*



is a commutative diagram of monics. Denote by $(G/H, q)$ *the object and arrow that is the cokernel of* $k \circ h$ *and by* $(K/H, p)$ *the object and arrow that is the cokernel of* h. *Then there exists a unique arrow* ϕ *so that*

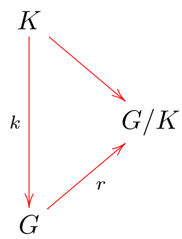


commutes.

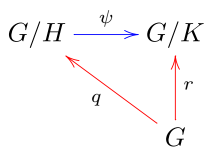
Proof. This is left as an exercise. □

Exercise 5.5.2. *Prove 5.5.1*

Lemma 5.5.3. *With notation as in 5.5.1, let*



be the cokernel diagram. Then, there exists a unique arrow ψ *so that*



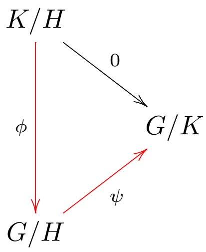
commutes.

Proof. Exercise. □

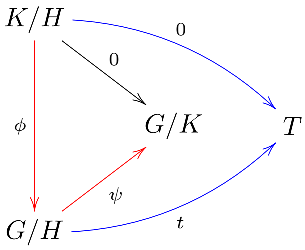
Exercise 5.5.4. *Prove 5.5.3.*

Theorem 5.5.5. *With notation as in 5.5.1 and 5.5.3, we have that* $G/K \simeq (G/H)/(K/H)$ *- the cokernel of* ϕ.

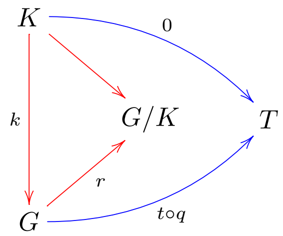
Proof. First note that



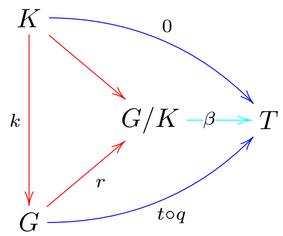
commutes. Now suppose that



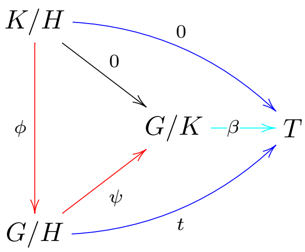
commutes. Then, since $t \circ \phi = 0$ and $\phi \circ p = q \circ k$, we have that $t \circ \phi \circ p = t \circ q \circ k = 0$. In other words,



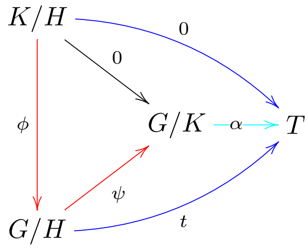
Thus, there exists a unique arrow β so that



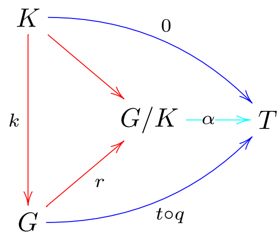
commutes. Now, $\beta \circ r = \beta \circ \psi \circ q = t \circ q$ which implies, since q is epi, that $\beta \circ \psi = t$. That is that



commutes. Suppose that α is an arrow so that



commutes. This implies, though, that $\alpha \circ \psi \circ q = \alpha \circ r = t \circ q$. Which implies that α also makes



commute. However, we know that β is the unique such arrow. Thus we must have that $\alpha = \beta$. Thus, we have proved that G/K and ψ satisfy the definition for the cokernel of ϕ we know that since cokernels are unique up to isomorphism, that $G/K \simeq (G/H)/(H/K)$. Which is what we claimed. □

5.6 Generators

The properties of **AbG** which we have discussed so far do not uniquely determine it. In addition to having Hom sets with an additive structure, a zero object, and the fact that all monics and epis are regular, the category of Abelian groups possesses something called a generator.

Definition 5.6.1. *Let* **C** *be a category. An object G of* **C** *is called a generator for* **C** *if for any pair of arrows in* **C**,

$$B \underset{v}{\overset{u}{\rightrightarrows}} C$$

and so that for every arrow $g : G \to B$, $u \circ g = v \circ g$ then it must follow that $u = v$.

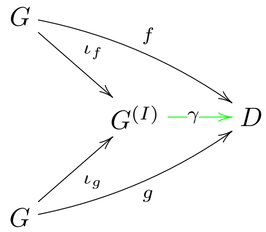
Exercise 5.6.2. *Show that in the category* **SET** *the one point set is a generator.*

Definition 5.6.3. *A category is said to be well-powered if for any object C and any set I, the sum $+_I C = C^{(I)}$ of $|I|$ copies of C exists.*

In particular, categories - like **AbG** - which have all colimits are well-powered.

Lemma 5.6.4. *Suppose that G is a generator in a category* **C**, *which is well-powered. Let D be an object in* **C**. *Then there exists a set I and an epic arrow $\gamma : G^{(I)} \to C$.*

Proof. Let $|Hom(G, D)| = |I|$ where $|X|$ is the cardinality of the set X. Let $f, g \in Hom(G, D)$. Let γ be the unique map so that



commutes. Suppose that $u, v : D \to W$ are two arrows so that $v \circ \gamma = u \circ \gamma$. Then, for example, $u \circ \gamma \circ \iota_f = u \circ f = v \circ \gamma \circ \iota_f = v \circ f$. Since f is an arbitrary arrow with source G, the definition of generator implies that $u = v$. Thus, γ is epic. □

What this means is that every Abelian group is a quotient object of some sum of the integers $\mathbb{Z}$.

5.7 Examples

We have already asserted that the integers $\mathbb{Z}$ are an object - a centrally important object - in the category **AbG**. Let's list other examples and briefly explore the category of **AbG** through a more traditional lens. First $\mathbb{Z}$ has a great many quotient groups, each defined by a function we shall label n with domain $\mathbb{Z}$ and range $\mathbb{Z}$ which sends $z \mapsto nz$. As a direct consequence of the definition of the *hom* sets in **AbG**, we must have for *any* arrow f with domain $\mathbb{Z}$ that $f(z + w) = f(z) + f(w)$.

Chapter 6

Commutative Rings

Traditionally, rings are thought of as Abelian groups with some additional structure - a set with two binary operations. However, the category of commutative rings differs greatly from the category **AbG**. In this chapter we intend to define the definition of the category of commutative rings, which we shall denote **CR**, by imposing increasingly restrictive conditions until we arrive at **CR**. Following Luo, there will be, broadly, 9 such conditions. As we did with **AbG**, upon the completion of these 9 sections we shall examine what sorts of objects - understood internally- live in **CR**.

6.1 Right Categories

As we saw in the last chapter, **AbG** has a zero object - that is, an object which is simultaneously initial and terminal. In the category of commutative rings **CR**, there is no such object. Like Abelian Groups, there is an object which, viewed traditionally consists of a set with one element, which we also call the zero ring. Specifically, the category of commutative rings has what is often called a "strict terminal object":

Definition 6.1.1. *A terminal object* 1 *in a category* $\mathbf{C}$ *is called strict if every arrow with* 1 *as its source is an isomorphism.*

The dual notion is - not surprisingly, perhaps - called a strict initial object.

Exercise 6.1.2. *Prove that any category which has both a strict initial and strict terminal object is a category with only one object.*

Exercise 6.1.3. *Prove that in* **SET***, the empty set is a strict initial object.*

Categories with strict initial objects are called left categories and those with strict terminal objects right categories. **CR** is a right category. (If not the right category).

6.2 Unitary Categories

Recall that we called a monic regular if there were a pair of arrows for which it was the equalizer. We shall also call the dual "regular" but shall be careful to differentiate between the two by always identifying an arrow as a regular *monic* or regular *epic*. To emphasize we state the following:

Definition 6.2.1. *An arrow is called a regular epic if it is a coequalizer for some pair of arrows.*

This terminology helps us identify a second defining characteristic of the category **CR**:

Definition 6.2.2. *A category is called right unitary*

1. *if it is a right category and*

2. *if every arrow $t : T \to 1$ to the strict terminal object is a regular epi*

Exercise 6.2.3. *The dual of a right analytic category is called left unitary. Show that the category of* **SET** *is left unitary.*

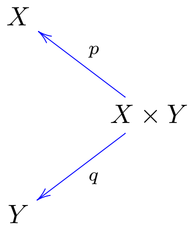
6.3 Extensive Categories

The dual category of **CR** shares properties with categories whose objects are best understood geometrically. **SET** is such a category. You have probably used Venn diagrams to understand certain facts about sets: the intersection, union and complement. For example, if the intersection - the pullback - of two sets is empty this can be expressed by showing two blobs with no overlap. Another way of saying this is to say that the pullback is initial. For those familiar with the category of topological spaces or of manifolds, the same is true. Objects understood geometrically have pullback equal to the strict initial object exactly when those objects do not overlap. We hope that this motivates the following:

Definition 6.3.1. *Two arrows with common source $b : A \to B$ and $c : A \to C$ are said to be codisjoint if their pushforward exists and is the strict terminal object.*

We chose the term "codisjoint" rather than "disjoint" here since disjoint is a term with clear geometric conotations and so would be most suggestive of that in the dual category to **CR**. As with other concepts the dual is indicated by use of the prefix "co". Finite products exist in **CR** and possess an interesting property:

Definition 6.3.2. *We will say that a product of objects*



is codisjoint *if the arrows p and q are codisjoint.*

The next four exercises could have been in chapter 3. Together, though, they will help illustrate some of the concepts we will be naming in this chapter.

Exercise 6.3.3. *Let*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

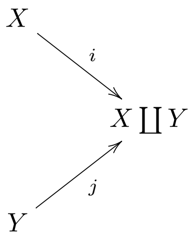
be functions. Show that the relation on Y, $y_1 \sim y_2$ if and only if there exists $x \in X$ so that $f(x) = y_1$ and $g(x) = y_2$ is an equivalence relation.

Exercise 6.3.4. *Let notation and setting be as in 6.3.3. Let D be the set whose elements are the equivalence classes defined on Y. Let $d : Y \to D$ be the function which sends each element to its equivalence class. Show that*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{d} D$$

is a coequalizer diagram

Exercise 6.3.5. *Let*



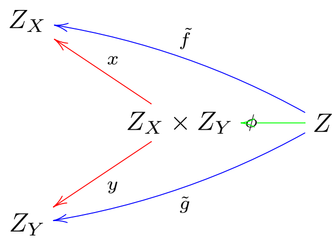
be a coproduct diagram. Let $f : Z \to X$ and $g : Z \to Y$ be arrows. Show that the pushforward of f and g is the coequalizer of $i \circ f$ and $j \circ g$.

Exercise 6.3.6. *Recall that the coproduct of two sets X and Y is their disjoint union. Use this fact and the last three exercises to construct the pushforward in* **SET**.

We shall find it easier to use the following notation when making the next definition:

Notation 6.3.7. *Let $f : X \times Y \to Z$ be an arrow. By Z_X we shall mean the pushforward of f by p the projection arrow in the product diagram above.*

Definition 6.3.8. *We keep the notation from 6.3.7. A category is said to have* costable products *if the unique arrow ϕ so that*



commutes - where $\tilde{f}$ is the pushforward of f by p and $\tilde{g}$ the pushforward of g by q, and where the red diagram is the product diagram - is an isomorphism.

The dual notion in this case is referred to as stable sums. Most geometric categories have them:

Exercise 6.3.9. *Show that* **SET** *has stable sums.*

Definition 6.3.10. *A right category is said to be right extensive if every product is codisjoint and costable.*

If you have successfully worked the exercises in this section, or else are willing to take them on faith, you have proven that **SET** is left extensive. As promised we are moving toward a definition of what **CR** is. So far we can comprehensibly say it is right extensive.

6.4 Rextensive Categories

Recall that we say a limit or colimit is "finite" if the number of objects in the limit or colimit diagram is finite.

Definition 6.4.1. *A rextensive category is a right extensive category in which all finite colimits exist.*

6.5 Analytic Categories

Recall that in the chapter on Abelian Groups we proved that every arrow can be uniquely factored - up to isomorphism - into the composition of a regular mono and an epi. Although this is not possible in every category, it is in **CR**. We make the following more general definition:

Definition 6.5.1. *We will refer to a category as right analytic if it is rextensive and if every arrow can be written as the composition of a regular epi followed by a monic.*

As you might have guessed, the dual notion is called left analytic. Many familiar and important categories are left analytic.

Exercise 6.5.2. *Prove that* **SET** *is left analytic. You many assume that it is lextensive - that is that its dual is rextensive.*

6.6 Right Analytic Geometries

This is the section where things get a bit more interesting. Recall that in reference? we showed that the pullback of a monic is monic - and, thus, by duality - showed that the pushforward of an epic is epic. But what about the reverse? Is the pullback of a epic epic? The pushforward of a monic monic? The answer is not always. Is it ever? The answer is - at least in **CR** - when the pushforward is by a special type of arrow.

Definition 6.6.1. *An arrow is said to be pre-flat provided that the pushforward of any mono by it is again a mono.*

The dual notion is called pre - coflat.

Exercise 6.6.2. *Find an example of an arrow in* **SET** *which is not pre - coflat.*

Exercise 6.6.3. *Show that the identity arrow is always pre-flat.*

Exercise 6.6.4. *Show that in a rextensive category, any arrow with terminal target is not pre-flat.*

Definition 6.6.5. *An arrow is said to be flat if every pushforward of it bis pre-flat.*

Again the dual is referred to as coflat.

Exercise 6.6.6. *Show that in in* **SET***, if a function is pre-coflat, then it is also coflat.*

We make the next definition in order to minimize ambiguity.

Definition 6.6.7. *An arrow f is said to factor through g if there exists an arrow h so that $f = g \circ h$ or $f = h \circ g$.*

We need this definition now, in order to define the notion of complement. The idea of complement is somewhat akin to that of that of top elements. More precisely, we have:

Definition 6.6.8. *Let e be an epi. The complement of e, which we shall often denote e^c, is an arrow satisfying the following criteria:*

1. *e and e^c are codisjoint*
2. *Every arrow which is codisjoint with e factors through e^c.*

Exercise 6.6.9. *Suppose that e and d are codisjoint. Show that the factorization of d through e^c is unique.*

Exercise 6.6.10. *Show that any two complements of an epi are isomorphic.*

Definition 6.6.11. *An strong epic is said to be codisjunctable if its complement is flat.*

Definition 6.6.12. *Let* $\{f_i : A \to B_i\}_{i\in I}$ *be a collection of epic arrows and objects. Let the colimit of this diagram - consisting of an object* U *and arrows* $a : A \to U$ *and - for every* $i \in I$ *-* $b_i : B_i \to U$*. Then the object* U *and the arrow* a *will be referred to as cointersection of the diagram.*

Exercise 6.6.13. *Show that the arrow* a *in 6.6.12 is necessarily epi.*

The dual notion is called intersection.

Exercise 6.6.14. *Show that in* **SET***, the intersection of the collection of subsets and inclusion functions* $\{f_i : X_i \to Y\}_{i\in I}$ *is the actual intersection of the sets*

$$\bigcap_{i\in I} X_{i\in I}$$

.

Definition 6.6.15. *We will say that a category is locally codisjunctable if every strong epic can be written as the cointersection of codisjunctable epis.*

Definition 6.6.16. *A category is said to be perfect if every cointersection of strong epis exist.*

Exercise 6.6.17. *Need the cointersection of epis be epi?*

Exercise 6.6.18. *Show that* **SET** *is perfect.*

Exercise 6.6.19. *Show that* **AbG** *is perfect.*

Definition 6.6.20. *An arrow is said to be nilpotent if the only arrow with which it is codisjoint is one whose target is the terminal object.*

Exercise 6.6.21. *Show that the identity arrow for any non terminal object is nilpotent.*

Definition 6.6.22. *The dual notion to nilpotent is called unipotent. That is, an arrow is unipotent if the only arrow with which it is disjoint is one whose source is the initial object.*

Exercise 6.6.23. *Show that in* **SET** *a function is unipotent if and only if it is onto.*

Definition 6.6.24. *An object is said to be reduced if any nilpotent arrow to it is monic.*

The fact that such objects will be of interest to us is, in part, what distinguishes **CR** from **SET**. Indeed, we have:

Exercise 6.6.25. *Show that there every object in* **SET** *is reduced.*

Definition 6.6.26. *A category is said to be reducibled if every non-terminal object has a reduced quotient object.*

Obviously **SET** is a perfect. Based on what we know about **AbG**, is it too perfect?

Definition 6.6.27. *A right analytic category which is locally codisjunctable reducible and perfect is called a right analytic geometry*

6.7 Coherent Analytic Categories

In the last chapter we briefly touched on objects which could be "finitely generated". We now define a similar notion, but in a disimilar fashion. In the last section we considered a particular type of limit - the intersection. Of course, every universal and couniversal we have considered can be regarded as a limit or colimit of a particular diagram. We now consider yet another sort of diagram which plays an important role in many applications, including those we shall consider in this text. For purposes of notational convenience we shall denote by $\mathcal{D}$ a diagram and include in this both the objects and the arrows. So, for example, if we wished to describe "pullback" as the limit of $\mathcal{D}$, then $\mathcal{D}$ would refer to the objects and arrows

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle x} \\ Y & \xrightarrow[y]{} & Z \end{array} .$$

As a matter of convention, we will always assume that a diagram contains all identity arrows of the objects in it.

Definition 6.7.1. *A diagram $\mathcal{D}$ is said to be filtered if*

1. *$\mathcal{D}$ is non-empty*

2. *if for every two objects A and B in $\mathcal{D}$, there exists an object C and arrows $a : C \to A$ and $b : C \to B$ in $\mathcal{D}$.*

3. *Given a pair of arrows $d, e : D \to E$ in $\mathcal{D}$, there exists an arrow $f : E \to F$ in $\mathcal{D}$ so that $f \circ e = f \circ d$.*

In this case, the colimit of such a diagram is called the filtered colimit. The dual notion is called the cofiltered limit. We may also permute these refering to cofiltered colimit and filtered limit

Definition 6.7.2. *We will say that a diagram $\mathcal{D}$ is a finite diagram if the numbers of both objects and arrows which it contains is finite.*

Exercise 6.7.3. *Let $\mathcal{D}$ be filtered diagram with a single object. Show that all arrows in $\mathcal{D}$ are isomorphisms.*

Lemma 6.7.4. *Let $\mathcal{D}$ be a finite non-empty cofiltered diagram containing objects $\{D_i\}1 \leq i \leq n$. Then there exists an object $D \in \mathcal{D}$ and arrows $\{\nu_i : D_i \to D\}_{1 \leq i \leq n}$.*

Proof. We proceed by induction on n:

1. $n = 1$: The condition is satisfied trivially, since we have imposed the convention that the identity arrow of each object is included in the diagram.

2. Suppose the claim holds for $n = k-1$. Suppose $\mathcal{D}$ contains k objects. By hypothesis, there exists an object $D \in \mathcal{D}$ and arrows $\{\nu_i : Di \to D\}_{1 \leq i \leq k1}$. Since $\mathcal{D}$ is filtered, there exists an object C and arrows $\eta : D \to$ and $\zeta : D_k \to C$. Thus, the arrows $\{\eta \circ \nu_i : Di \to C, \zeta : D_k \to C\}_{1 \leq i \leq k1}$ satisfy the claim made in this lemma.

□

Lemma 6.7.5. *Suppose $\mathcal{D}$ is a filtered diagram. Let $\{q_i : C \to D\}_{1 \leq i \leq n}$ be a collection of arrows in $\mathcal{D}$. Then, there exists an object $E \in \mathcal{D}$ and arrow $e : D \to E$ so that for all $1 \leq i, j \leq n$, $e \circ q_i = e \circ q_j$.*

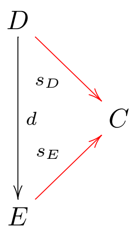
Proof. We proceed by induction on n:

1. Suppose $n = 1$. Then, the commutativity of the diagram
$$C \xrightarrow{q_1} D \xrightarrow{1_D} D4$$
confirms the lemma.

2. Suppose the lemma holds for $n = k-1$. Then, there exists an arrow $z : D \to E$ so that $z \circ q_i = z \circ q_j$ for all $1 \leq i, j \leq k-1$. Since $\mathcal{D}$ is filtered, there exists an arrow in it, $t : E \to F$ so that $t \circ z \circ q_k = t \circ z \circ q_1$. If$1 \leq i, j \leq k-1$ Then $t \circ z \circ q_i = t \circ z \circ q_j$. Thus, if $1 \leq i, j \leq k$, $t \circ z \circ q_i = t \circ z \circ q_j$. This is what the lemma claimed.

□

Lemma 6.7.6. *Let $\mathcal{D}$ be a finites cofiltered diagram. Then there exists an object C and arrows $\{s_D : C \to D\}_{d \in \mathcal{D}}$ so that*



commutes for every arrow $d \in \mathcal{D}$.

Proof. Let D be an arbitrary object in $\mathcal{D}$.By 6.7.4 there exists an object $F \in \mathcal{D}$ and an arrow $f_D : D \to F$. Let $d : E \to G$ be an arrow in $\mathcal{D}$. Then, by **??** there exists an object C_d and arrow $g_d : F \to C_d$ so that $g_d \circ f_G \circ d = g_d \circ f_E$. By assumption, there are finitely many such d, so 6.7.4 again implies that there exists an object H and arrows $h_d : C_d \to H$, while **??** implies that there exists an object K and arrow $k : H \to K$ so that if $b : B \to A$ is another arrow in $\mathcal{D}$, $k \circ h_d \circ g_d = k \circ h_b \circ g_b$. Let $l_D = k \circ h_d \circ g_d \circ f_D$. Then the collection of arrows l_D and object K satisfy the claim in the lemma. □

To get a better handle on this definition let's prove that a fairly familiar collection is filtered:

Lemma 6.7.7. *Let X be a set. Then the collection of subsets of X, $\mathcal{P}(X)$ together with the the inclusion functions is filtered.*

Proof. We take each of the 3 criteria in 6.7.1 in turn.

1. Since $\emptyset \in \mathcal{P}$, $\mathcal{P} \neq \emptyset$.
2. Let $A, B \in \mathcal{P}$. Then $A \bigcap B \subseteq A$ and $A \bigcap B \subseteq B$, thus, by definition of our diagram, there are functions from $A \bigcap B$ to both A and B.
3. Is satisfied trivially, since there are no distinct parallel arrows.

□

Recall that our definition of category requires that given two objects A and B of a category **C**, the collection of $Hom(A, B)$ is a set. Suppose, as above, that $\mathcal{D}$ is a diagram in a category **C**, and that $\mathcal{D}$ is filtered. Letus establish the following notation:

Notation 6.7.8. *Let $\mathcal{D}$ be a diagram. We shall sometimes write*

$$\lim\_ \rightarrow \mathcal{D}$$

for the colimit of $\mathcal{D}$, and

$$\lim_{\leftarrow} \mathcal{D}$$

for the limit of $\mathcal{D}$.

Definition 6.7.9. *An object K in a category **C** is said to be finitely presented if for every filtered diagram $\mathcal{D}$ of **C**, the two sets*

$$Hom(K, \lim_{\rightarrow} \mathcal{D})$$

and colimit$Hom(K, D)$ are the same.

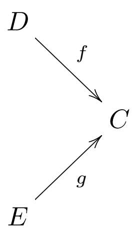
What does this mean? In partial answer to this question let's examine what the colimit of a filtered diagram in **SET** looks like.

Lemma 6.7.10. *Let $\mathcal{D}$ be a filtered diagram of sets and functions between them. Then, colim$\mathcal{D}$ exists and has as its object the set of equivalence classes on*

$$\coprod_{D \in \mathcal{D}} D$$

defined by

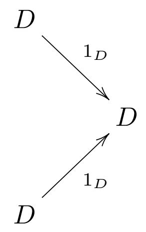
$$d \sim e \Leftrightarrow \exists$$



with $f(d) = g(e)$. It has as its arrows the functions $i : D \rightarrow colim\mathcal{D}$ defined by $d \mapsto \bar{d}$ where $\bar{d}$ is the equivalence class to which d belongs.

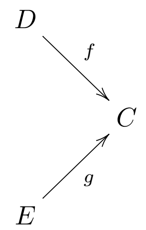
Proof. First, we need to prove that our claimed equivalence relation *really is* and equivalence relation. Let's check each of the requirements in turn:

1. (Reflexivity) Let $d \in \coprod \mathcal{D}$. Then there exists $D \in \mathcal{D}$ with $d \in D$. Clearly, in the diagram



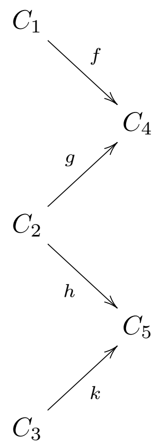
   we have $1_D(d) = 1_D(d)$

2. (Symmetry) Suppose $d \in D$ and $e \in E$ for objects $D, E \in \mathcal{D}$, and that $d \sim e$. Then there exists

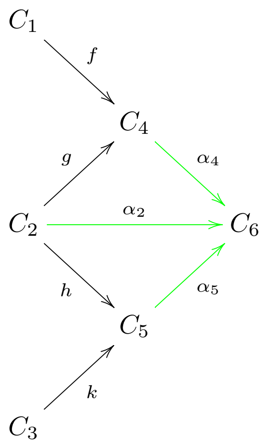


   with $f(d) = g(e)$. Then, $e \sim d$.

3. (Transitivity) Suppose $x \in C_1$, $y \in C_2$ and $z \in C_3$ and that $x \sim y$ and $y \sim z$. Then, there exists objects C_4, and C_5 and functions f, g, h, and k all in $\mathcal{D}$ so that in the diagram



   $f(x) = g(y)$ and $h(y) = k(z)$. Since $\mathcal{D}$ is filtered and the above diagram is finite, 6.7.6 implies that there exists an object C_6 and functions α_i in $\mathcal{D}$ so that

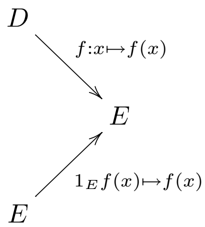


commutes. Then, however, we have that

$$\alpha_4 \circ f(x) = \alpha_4 \circ g(y) = \alpha_2(y) = \alpha_5 \circ h(y) = \alpha_5 \circ k(z).$$

That is, $x \sim z$, as required.

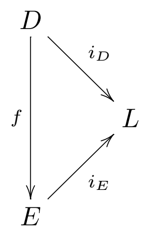
Now, let $d : D \to E$ be an arrow in a filtered diagram $\mathcal{D}$ in SET. Let C be the collection of equivalence classes of the coproduct (which is the disjoint union in SET) of the objects in $\mathcal{D}$. Let $x \in D$ be arbitray and i_D and i_E be the functions described in the statement of this lemma. Since



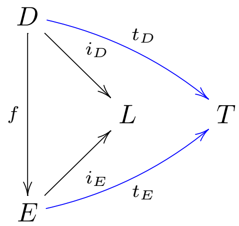
satisfies the criteria we laid our in our definition of $\sim$, we see that $x \sim f(x)$. Thus,

$$i_E(f(x)) = f\bar{(}x) = \bar{x} = i_D(x)$$

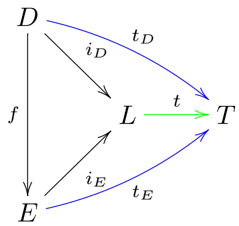
Thus, for every such f, the diagram



commutes. Suppose



is a commutative diagram. Define $t : L \to T$ by $\bar{x} \mapsto t_D(x)$ for any $x \in D$. I claim many things:
The function is well-defined Suppose $x \sim y$ for some $y \in K$. Then, by definition, there exists functions $g : D \to G$ and $h : K \to G$ and a set G all in $\mathcal{D}$ so that, $g(x) = h(y)$. Since T and its arrows makes the original diagram commute, $t_D(x) = t_G(g(x)) = t_G(h(y)) = t_G(y)$.



commutes by definition of t.
It is Unique. Why? □

Exercise 6.7.11. *Prove the last Why? in the proof.*

Exercise 6.7.12. *Prove the statement Then, $e \sim d$ in the proof above.*

Definition 6.7.13. *A category is said to be locally finitely presentable if all all limits exist and if every object is the cointersection of finitely presentable objects.*

Up until now we have hinted at a relationship between finitely generated and finitely presented. Just as sets have subsets and objects have subobjects, categories have subcategories.

Definition 6.7.14. *Let* **C** *be a category. Then* **S** *is said to be a subcategory of* **C** *if* **S** *is a category all of whose objects and arrows are objects and arrows of* **C**.

We now put these last two notions - finitely presented objects and subcategory - together to finally yield the definition of a right coherent analytic category.

Definition 6.7.15. *A right coherent analytic category. is a locally finitely presentable category whose subcategory of finitely presentable objects is rextensive.*

6.8 Stone Geometries

Remember that a product of a collection of objects, like any limit, consists of both an object and collection of arrows. These arrows are interesting in their own right and so have a separate name.

Definition 6.8.1. *Let* $\{D_i\}$ *be a collection of objects in a category and* $\langle P, p_i \rangle$ *its product. Then the arrows* p_i *are called direct*

Definition 6.8.2. *A category is called a Right Stone Geometry if it is right coherent analytic and if any strong epi is the cointersection of direct epis.*

The dual of a right stone geometry is called a left Stone geometry.

Exercise 6.8.3. *Is* **SET** *a left Stone geometry?*

Exercise 6.8.4. *Let* **POS** *be the category whose objects are partially ordered sets and whose arrows are functions which preserve the order on them. That is* $f : P \to Q$ *is an arrow in* **POS** *if* $a \leq b \Rightarrow f(a) \leq f(b)$*. Is* **POS** *a Stone Geometry?*

6.9 Right Coherent Analytic Geometries

Definition 6.9.1. *A category is said to be a right coherent analytic geometry if it is locally codisjunctable and a right coherent analytic category.*

In the sections which follow we shall take the category of commutative rings **CRng** to be a subcategory of a right coherent analytic geometry. In particular we shall declare that **CRng** is a right coherent analytic geometry. However, I do not know whether the converse of this declaration is so: That being a right coherent analytic geometry characterizes **CRng**.

6.10 The Lattice of Strong Quotient Objects

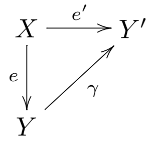
In our study of **AbG** we found that the collection of subobjects of a given object formed a partially ordered set. In fact, this collection formed a lattice. The same is true for what I refer to as “strong quotient objects” a term we can formally define.

Definition 6.10.1. *Let* X *be an object. We will say that* Y *is a strong quotient object of* X *if there exists a strong epi* $e : X \to Y$*.*

Recall that part of the definition of **CRng** is that it has regular epi monic factorizations. In other words every arrow f in **CRng** can be factored $f = m \circ e$ where e is a regular epi. Recall too that regular implies strong. All together this leads to the following definition:

Definition 6.10.2. *Let $f : X \to Y$ be an arrow in a right analytic category. Then we shall call the target of the regular epi in the regular epi monic factorization of f the strong image of f.*

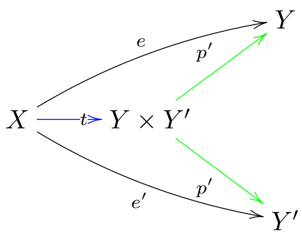
Thus, for every arrow with source X in **CRng**, there is a strong quotient object - namely the strong image of that arrow. It is possible to define a partial order on these strong quotient objects. Specifically we will say that if $e : X \to Y$ and $e' : X \to Y'$ are two strong quotient objects of X then $\langle Y, e\rangle \leq \langle Y', e'\rangle$ provided there exists an epi γ so that



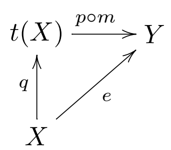
commutes.

Lemma 6.10.3. *Let X be an object in* **CRng**. *Then the meet of any two strong quotient objects of X exists in the partial order defined above.*

Proof. Let $\langle Y, e\rangle$ and $\langle Y', e'\rangle$ be quotient objects of X. Then, there exists a unique arrow t so that



commutes. Denote by $q : X \to t(X)$ the strong image of $t = m \circ q$. First note that since $p \circ t = p \circ m \circ q = e$, $p \circ m$ must be epi. Similarly, $p' \circ m$ is also epi. Put another way, we have the commutative diagrams



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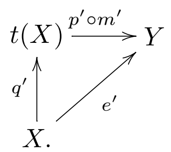
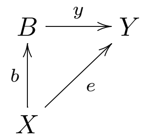


Put still another way, $\langle t(X), q\rangle \leq \langle e, Y\rangle$ and $\langle t(X), q\rangle \leq \langle e', Y'\rangle$. So, $\langle t(X), q\rangle$ is a lower bound for both $\langle e, Y\rangle$ and $\langle e', Y'\rangle$.Now suppose that $\langle B, b\rangle$ is also a lower bound for both $\langle e, Y\rangle$ and $\langle e', Y'\rangle$. Then we have commutative diagrams



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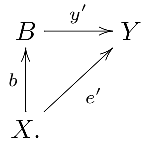
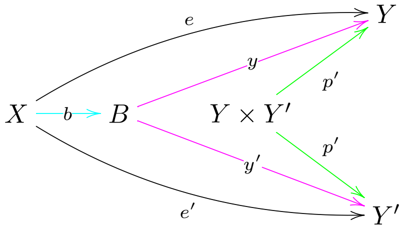
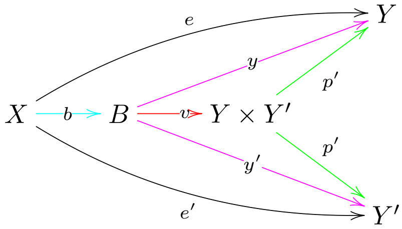


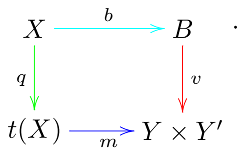
where y and y' are epis. Thus, we have the commutative diagram



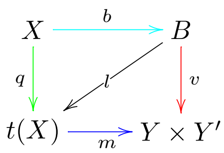
There also exists a uniqe v so that



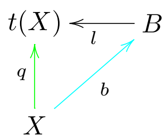
commutes. But, this implies that $t = m \circ q = v \circ b$.In other words, we have the commutative diagram



Remember, though, that q is strong. Thus, by **??**, there exists a unique l so that in the diagram



we have that



commutes. In other words, $\langle t(X), q\rangle \leq \langle B, b\rangle$. That is $\langle t(X), q\rangle$ is the *greatest* of all lower bounds. □

Exercise 6.10.4. *Prove the last statement in red in the proof.*

We now prove that any two strong epis of X have a greatest lower bound.

Lemma 6.10.5. *Let X be an object in* **CRng**. *Then the join of any two strong quotient objects of X exists in the partial order defined above.*

Proof. Exercise. □

Exercise 6.10.6. *Prove 6.10.5. Hint: Recall that the glb of two monics with common source is their pullback and that the pushforward of a strong epi is again a strong epi.)*

6.11 Objects of Interest

Definition 6.11.1. *An object in* **CRng** *will be called irreducible if the join of any two of its strong quotient objects is not the terminal object.*

Definition 6.11.2. *An object in* **CRng** *will be called primary if every analytic epic from it is monic.*

Exercise 6.11.3. *Show that every subobject of a primary object is primary.*

Definition 6.11.4. *An object in* **CRng** *will be called integral if it is both primary and reduced.*

Exercise 6.11.5. *Show that a subobject of an integral object is also integral*

Exercise 6.11.6. *Show that every integral object is also irreducible.*

Definition 6.11.7. *An object in* **CRng** *will be called a fieldif every arrow which has it as its source is monic.*

Exercise 6.11.8. *Show that any field is nilpotent*

Exercise 6.11.9. *Show that if F is a field than every strong epi $e : F \to R$ is an isomorphism*

Exercise 6.11.10. *Show that every field is irreducible.*

Definition 6.11.11.

Chapter 7

Functors

Part of the philosophy of category theory is that the essential information about a mathematical object is carried - in fact defined by - properties of the arrows to and from that object. This philosophy is applied not just within categories, but among them as well.

Definition 7.0.12. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two categories. A covariant functor* F *between* $\mathbf{C}$ *and* $\mathbf{D}$ *is a pair of functions which will both be denoted* F*. The first assigns objects* X *of* $\mathbf{C}$ *to objects* FX *in* $\mathbf{D}$*. The second assigns arrows in* $\mathbf{C}$ f *to arrows* Ff *in* $\mathbf{D}$ *in such a way that*

1. $Ff \circ g = Ff \circ Fg$
2. $F1_X = 1_{FX}$

for all objects X *and arrows* f *and* g *of* $\mathbf{C}$.

If $Ff \circ g = Fg \circ Ff$, then F is called a contravariant functor.

Example 7.0.13. *The assignment of a set to its power set defines a contravariant functor* $\mathcal{P}$ *from* **SET** *to* **SET***. If* $f : X \to Y$ *is a function then* $\mathcal{P}f : B \mapsto f^{-1}(B)$ *for every* $B \subseteq Y$.

Exercise 7.0.14. *Prove that* $\mathcal{P}$ *just defined in 7.0.13 is indeed a functor.*

Exercise 7.0.15. *Suppose* $F : \mathbf{C} \to \mathbf{D}$ *is a functor. Show that if* f *is an isomorphism, then so is* Ff.

Chapter 8

Fields

Chapter 9

Functors

Chapter 10

Modules