HW §4.6 Numbers 2,7,8,11,17

2.

1. $\{(1,1),(2,2),(3,3)\}$
2. $\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$
3. $\{(1,1),(2,2),(3,3)(1,3),(3,1)\}$
4. $\{(1,1),(2,2),(3,3),(2,3),(3,2)\}$
5. $\{(1,1),(2,2),(3,3)(1,2)(2,3),(1,3),(3,1),(3,2),(2,1)\}$

7.

Lemma 1. *Suppose A is a set and $\mathcal{F}$ is a partition of A. let*

$$R = \cup_{X \in \mathcal{F}}(X \times X)$$

Then, R is an equivalence relation on A.

Proof. We have already seen that R is reflexive.
Symmetry: Let $a \in A$ be arbitrary. Let $b \in A$ be arbitrary. Suppose $(a,b) \in R$. Then we can find some $X \in \mathcal{F}$ so that $(a,b) \in X \times X$. Then $a \in X$ and $b \in X$. Thus, $(b,a) \in X \times X$. Since $X \in \mathcal{F}$, $(b,a) \in \cup_{X \in \mathcal{F}}(X \times X)$. Thus, $(b,a) \in R$. Since a and b were arbitrary, R is symmetric.
Transitivity: Let $a \in A$ be arbitrary. Let $b \in A$ be arbitrary. Let $c \in A$ be arbitrary. Suppose $(a,b) \in R$. Suppose $(b,c) \in R$. Since $(a,b) \in R$, we can find $X \in \mathcal{F}$ so that $(a,b) \in X \times X$. Since $(b,c) \in R$, we can find $Y \in \mathcal{F}$ so that $(b,c) \in Y \times Y$. Suppose $X \neq Y$. Then, since $b \in X \cap Y$, $X \cap Y \neq \emptyset$ which is a contradiction to our supposition that $\mathcal{F}$ is an equivalence relation. Thus, $X = Y$. Thus, $(b,c) \in X \times X$. So, we have that $a \in X$, $b \in X$ and $c \in X$. Thus, $(a,c) \in X \times X$. Thus, $(a,c) \in R$. Since a, b and c were arbitrary, R is transitive.
Thus, since R is reflexive, symmetric and transitive it is an equivalence relation. □

8.

theorem 2. *Suppose R and S are equivalence relations on A and $A/R = A/S$. Then, $R = S$.*

Proof. Suppose R and S are equivalence relations on A and $A/R = A/S$. Let $a \in A$ be arbitrary. Let $b \in A$ be arbitrary. Suppose $(a,b) \in R$. Then, $b \in [a]_R$, thus $b \in [a]_S$, since $A/R = A/S$. Thus, $(a,b) \in S$. Since a and b were arbitrary, $R \subseteq S$.
Let $a \in A$ be arbitrary. Let $b \in A$ be arbitrary. Suppose $(a,b) \in S$. Then,

$b \in [a]_S$, thus $b \in [a]_R$, since $A/R = A/S$. Thus, $(a, b) \in R$. Since a and b were arbitrary, $S \subseteq R$.
Since $R \subseteq S$ and $S \subseteq R$, $R = S$. ☐

11.

theorem 3. *For every integer n, either $n^2 \equiv 0 \pmod{|}4|$ or $n^2 \equiv 1 \pmod{|}4|$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary.
<u>Case 1</u>:Suppose $n \equiv 0 \pmod{|}4|$. Then, we can find $k \in \mathbb{Z}$ so that $n = 4k$. Thus, $n^2 = 16k^2 = 4(4k^2)$, thus, $n^2 \equiv 0 \pmod{|}4|$.
<u>Case 2:</u> Suppose $n \equiv 1 \pmod{|}4|$. Then, we can find $k \in \mathbb{Z}$ so that $n = 4k + 1$. Thus, $n^2 = 16k^2 + 8k + 1 = 4(4K^2 + 2k) + 1$. Thus, $n^2 \equiv 1 \pmod{|}4|$.
<u>Case 3:</u> Suppose $n \equiv 2 \pmod{|}4|$. Then, we can find $k \in \mathbb{Z}$ so that $n = 4k + 2$. Thus, $n^2 = 16k^2 + 16k + 4 = 4(4K^2 + 4k + 1)$. Thus, $n^2 \equiv 0 \pmod{|}4|$.
<u>Case 2:</u> Suppose $n \equiv 3 \pmod{|}4|$. Then, we can find $k \in \mathbb{Z}$ so that $n = 4k + 3$. Thus, $n^2 = 16k^2 + 24k + 9 = 4(4K^2 + 6k + 2) + 1$. Thus, $n^2 \equiv 1 \pmod{|}4|$.
Since in each of these exhaustive cases, either $n^2 \equiv 0 \pmod{|}4|$ or $n^2 \equiv 1 \pmod{|}4|$, and since n was arbitrary, we conclude that for every integer n, either $n^2 \equiv 0 \pmod{|}4|$ or $n^2 \equiv 1 \pmod{|}4|$. ☐

17.

theorem 4. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are partitions of a set A. Define*

$$\mathcal{F} \cdot \mathcal{G} := \{Z \in \mathcal{P}(A) | Z \neq \emptyset and \exists X \in \mathcal{F} \exists Y \in \mathcal{G}(Z = X \cap Y)\}$$

Then, $\mathcal{F} \cdot \mathcal{G}$ is also a partition of A.

Proof. Suppose $\mathcal{F}$ and $\mathcal{G}$ are partitions of a set A. Define

$$\mathcal{F} \cdot \mathcal{G} := \{Z \in \mathcal{P}(A) | Z \neq \emptyset \text{ and } \exists X \in \mathcal{F} \exists Y \in \mathcal{G}(Z = X \cap Y)\}$$

<u>$\cup\mathcal{F} \cdot \mathcal{G} = A$</u>: Let x be arbitrary. Suppose $x \in \cup\mathcal{F} \cdot \mathcal{G}$. Then, we can find $Z \in \mathcal{P}(A)$ so that $x \in Z$. Thus, since $Z \subseteq A$ and $x \in Z$, $x \in A$. Thus, $\cup\mathcal{F} \cdot \mathcal{G} \subseteq A$
Let x be arbitrary. Suppose $x \in A$. Since $\mathcal{F}$ is a partition, we can find $X \in \mathcal{F}$ with $x \in X$. Since $\mathcal{G}$ is a partition, we can find $X \in \mathcal{G}$ with $x \in Y$. Thus, $x \in X \cap Y = Z$, and so $x \in \cup\mathcal{F} \cdot \mathcal{G}$. Thus, $\cup\mathcal{F} \cdot \mathcal{G} \supseteq A$. Thus, since $\cup\mathcal{F} \cdot \mathcal{G} \subseteq A$ and $\cup\mathcal{F} \cdot \mathcal{G} \supseteq A$, $\cup\mathcal{F} \cdot \mathcal{G} = A$.
<u>$\mathcal{F} \cdot \mathcal{G}$ is Pairwise Disjoint:</u> Let $W \in \mathcal{F} \cdot \mathcal{G}$ be arbitrary. Let $T \in \mathcal{F} \cdot \mathcal{G}$ be arbitrary. Then, we can find $A \in \mathcal{F}$ and $B \in \mathcal{G}$ so that $W = A \cap B$. Then, we can find $C \in \mathcal{F}$ and $D \in \mathcal{G}$ so that $T = C \cap D$. Suppose $W \neq T$. Let x be arbitrary. Suppose $x \in W \cap T$. Then, since $W = A \cap B$, $x \in A$ and $x \in B$. Then, since $T = C \cap D$, $x \in C$ and $x \in D$. Thus, $x \in A \cap C$ and $x \in B \cap D$. Since $\mathcal{F}$ is a partition, it is pairwise disjoint. Thus, $A = C$. Also, since $\mathcal{F}$ is a partition, it is pairwise disjoint. Thus, $B = D$. Thus, $W = A \cap B = C \cap D = T$ which is a contradiction. Thus, $x \notin W \cap T$. Thus, $W \cap T = \emptyset$. Since W and T were arbitrary, $\mathcal{F} \cdot \mathcal{G}$ is pairwise disjoint.

Finally, if $Z \in \mathcal{F} \cdot \mathcal{G}$, then $Z \neq \emptyset$, by definition.
We have thus shown that all three criteria for being a partition have been met.
Thus, $\mathcal{F} \cdot \mathcal{G}$ is a partition. $\square$