HW §4.4 Numbers 2,3,6,7,20

2.
a.
This is a partial order. In fact, it is a total order.
b.
This is not a partial order. For example, it is not anti-symmetric since exquisite$R$Escort and Escort$R$Exquisite, but Escort$\neq$Exquisite.
c.
This is a total order.

3.
a.
Minimal:$\{2\}$.
Maximal:$\{3, 4\}$.
Smallest:$\{2\}$.
Largest: none.
l.u.b of B:none.
g.l.b of B:$\{2\}$.
b.
Minimal:$\{1\}$.
Maximal:$\{2\}$.
Smallest:$\{1\}$.
Largest: none.
l.u.b of B:$\{2\}$
g.l.b of B:$\{1\}$.
c.
Minimal:$\emptyset$
Maximal:$\{a, b, c, d, e | a, b, c, d, e \in \mathbb{N}\}$.
Smallest:$\emptyset$.
Largest: $\mathbb{N}$
l.u.b of B:None
g.l.b of B:$\emptyset$.
6.
a.

theorem 1. *Suppose R_1 and R_2 are partial orders on A. Then so is $R_1 \cap R_2$.*

Proof. Suppose R_1 and R_2 are partial orders on A.
Reflexivity:Let $a \in A$ be arbitrary. Then, since R_1 is a partial order, $(a, a) \in R_1$. Similarly, since R_2 is a partial order, $(a, a) \in R_2$. Thus, $(a, a) \in R_1 \cap R_2$. Since $a \in A$ was arbitrary, $R_1 \cap R_2$ is reflexive.
Anti-Symmetry:Let $a \in A$ be arbitrary. Let $b \in A$ be arbitrary. Suppose $(a, b) \in R_1 \cap R_2$ and $(b, a) \in R_1 \cap R_2$. Then, $(a, b) \in R_1$ and $(b, a) \in R_1$. Since R_1 is anti-symmetric, $a = b$. Since a and b were arbitrary, $R_1 \cap R_2$ is anti-symmetric.
Transitivity:Let $a \in A$ be arbitrary. Let $b \in A$ be arbitrary. Let $c \in A$ be

arbitrary. Suppose $(a,b) \in R_1 \cap R_2$ and $(b,c) \in R_1 \cap R_2$. Thus, $(a,b) \in R_1$ and $(b,c) \in R_1$. Then, since R_1 is a partial order, and thus transitive, $(a,c) \in R_1$. Thus, $(a,b) \in R_2$ and $(b,c) \in R_2$. Then, since R_2 is a partial order, and thus transitive, $(a,c) \in R_2$. Thus, $(a,c) \in R_1 \cap R_2$. Since a, b, and c were arbitrary, $R_1 \cap R_2$ is tranistive.
Since $R_1 \cap R_2$ is reflexive, anti-symmetric and transitive it is a partial order. ☐

b.
$R_1 \cup R_2$ is not necessarily a partial on A. For example, let $A = \{a,b,c\}$, $R_1 = \{(a,a),(b,b),(c,c),(a,b)\}$ and $R_2 = \{(a,a),(b,b),(c,c),(b,c)\}$. Then, R_1 and R_2 are partial orders. Note, though, that while $(a,b) \in R_1 \cup R_2$ and $(b,c) \in R_1 \cup R_2$, $(a,c) \notin R_1 \cup R_2$ and so $R_1 \cup R_2$ is not transitive and so not a partial order.

7.
a.

theorem 2. *Suppose that R_1 is a partial order on A_1 and R_2 is a partial order on A_2. Suppose $A_1 \cap A_2 = \emptyset$. Then, $R_1 \cup R_2$ is a partial order on $A_1 \cup A_2$.*

Proof. Suppose that R_1 is a partial order on A_1 and R_2 is a partial order on A_2. Suppose $A_1 \cap A_2 = \emptyset$.
Reflexivity: Let $a \in A_1 \cup A_2$ be arbitrary.

Case 1:Suppose $a \in A_1$. Then, since R_1 is a partial order on A_1, $(a,a) \in R_1$, thus $(a,a) \in R_1 \cup R_2$.

Case 2:Suppose $a \in A_2$. Then, since R_2 is a partial order on A_2, $(a,a) \in R_2$, thus $(a,a) \in R_1 \cup R_2$.

Since in any case $(a,a) \in R_1 \cup R_2$ and since a was arbitrary, $R_1 \cup R_2$ is reflexive.
Anti-Symmetry: Let $a \in A_1 \cup A_2$ be arbitrary. Let $b \in A_1 \cup A_2$ be arbitrary. Suppose $(a,b) \in R_1 \cup R_2$. Suppose$(b,a) \in R_1 \cup R_2$.

Case 1: Suppose $(a,b) \in R_1$. Then, both a and b are elements of A_1. Since $A_1 \cap A_2 = \emptyset$, neither a nor b are elements of A_2. Thus, $(b,a) \in R_1 \cup R_2$ implies that $(b,a) \in R_1$. Since R_1 is a partial order and $(a,b) \in R_1$ and $(b,a) \in R_1$, $a = b$. Since a and b were arbitrary, in this case,$R_1 \cup R_2$ is anti-symmetric.

Case 2: Suppose $(a,b) \in R_2$. Then, both a and b are elements of A_2. Since $A_1 \cap A_2 = \emptyset$, neither a nor b are elements of A_1. Thus, $(b,a) \in R_1 \cup R_2$ implies that $(b,a) \in R_2$. Since R_2 is a partial order and $(a,b) \in R_2$ and $(b,a) \in R_2$, $a = b$. Since a and b were arbitrary, in this case,$R_1 \cup R_2$ is anti-symmetric.

Thus, in either of these two exhaustive cases, $R_1 \cup R_2$ is anti-symmetric.
Transitivity: Let $a \in A_1 \cup A_2$ be arbitrary. Let $b \in A_1 \cup A_2$ be arbitrary. Let $c \in A_1 \cup A_2$ be arbitrary.Suppose $(a,b) \in R_1 \cup R_2$. Suppose$(b,c) \in R_1 \cup R_2$.

Case 1: Suppose $(a,b) \in R_1$. Then, both a and b are elements of A_1. Since $A_1 \cap A_2 = \emptyset$, neither a nor b are elements of A_2. Thus, in particular, $(b,c) \notin R_2$.Thus, $(b,c) \in R_1 \cup R_2$ implies that $(a,c) \in R_1$. Since R_1 is a partial order and $(a,b) \in R_1$ and $(b,c) \in R_1$, $(a,c) \in R_1$. Since a, b and c were arbitrary, in this case,$R_1 \cup R_2$ is transitive.

Case 2: Suppose $(a, b) \in R_2$. Then, both a and b are elements of A_2. Since $A_1 \cap A_2 = \emptyset$, neither a nor b are elements of A_1. Thus, in particular, $(b, c) \notin R_1$.Thus, $(b, c) \in R_1 \cup R_2$ implies that $(b, c) \in R_2$. Since R_2 is a partial order and $(a, b) \in R_2$ and $(b, c) \in R_2$, $(a, c) \in R_2$. Since a, b and c were arbitrary, in this case,$R_1 \cup R_2$ is transitive.
Thus, in either of these two exhaustive cases,$R_1 \cup R_2$ is transitive.
Since $R_1 \cup R_2$ is reflexive, anti-symmetric and transitive, it is a partial order. $\square$

b.

theorem 3. *Suppose that R_1 is a partial order on A_1 and R_2 is a partial order on A_2. Suppose $A_1 \cap A_2 = \emptyset$. Then, $R_1 \cup R_2 \cup (A_1 \times A_2)$ is a partial order on $A_1 \cup A_2$.*

Proof. Suppose that R_1 is a partial order on A_1 and R_2 is a partial order on A_2. Suppose $A_1 \cap A_2 = \emptyset$.
Reflexivity: Let $a \in A_1 \cup A_2$ be arbitrary.

Case 1:Suppose $a \in A_1$. Then, since R_1 is a partial order on A_1, $(a, a) \in R_1$, thus $(a, a) \in R_1 \cup R_2 \cup (A_1 \times A_2)$.

Case 2:Suppose $a \in A_2$. Then, since R_2 is a partial order on A_2, $(a, a) \in R_2$, thus $(a, a) \in R_1 \cup R_2 \cup (A_1 \times A_2)$.
Since in any case $(a, a) \in R_1 \cup R_2 \cup (A_1 \times A_2)$ and since a was arbitrary, $R_1 \cup R_2$ is reflexive.
Anti-Symmetry: Let $a \in A_1 \cup A_2$ be arbitrary. Let $b \in A_1 \cup A_2$ be arbitrary. Suppose $(a, b) \in R_1 \cup R_2 \cup (A_1 \times A_2)$. Suppose$(b, a) \in R_1 \cup R_2 \cup (A_1 \times A_2)$.

Case 1: Suppose $(a, b) \in R_1$. Then, both a and b are elements of A_1. Since $A_1 \cap A_2 = \emptyset$, neither a nor b are elements of A_2. Thus, $(b, a) \in R_1 \cup R_2 \cup (A_1 \times A_2)$ implies that $(b, a) \in R_1$. Since R_1 is a partial order and $(a, b) \in R_1$ and $(b, a) \in R_1$, $a = b$. Since a and b were arbitrary, in this case,$R_1 \cup R_2$ is anti-symmetric.

Case 2:Suppose $(a, b) \in R_2$. Then, both a and b are elements of A_2. Since $A_1 \cap A_2 = \emptyset$, neither a nor b are elements of A_1. Thus, $(b, a) \in R_1 \cup R_2 \cup (A_1 \times A_2)$ implies that $(b, a) \in R_2$. Since R_1 is a partial order and $(a, b) \in R_2$ and $(b, a) \in R_2$, $a = b$. Since a and b were arbitrary, in this case,$R_1 \cup R_2$ is anti-symmetric.
Case 3: Suppose $(a, b) \in A_1 \times A_2$. Then, $a \in A_1$ and $b \in A_2$. But then (b, a) cannot be in $R_1 \cup R_2 \cup (A_1 \times A_2)$. Thus, in this case, $R_1 \cup R_2 \cup (A_1 \times A_2)$ is anti-symmetric.
Since in each of these three exhaustive cases, $R_1 \cup R_2 \cup (A_1 \times A_2)$ is anti-symmetric, we have proved that $R_1 \cup R_2 \cup (A_1 \times A_2)$ is anti-symmetric.
Transitivity: Let $a \in A_1 \cup A_2$ be arbitrary. Let $b \in A_1 \cup A_2$ be arbitrary. Let $c \in A_1 \cup A_2$ be arbitrary.Suppose $(a, b) \in R_1 \cup R_2 \cup (A_1 \times A_2)$. Suppose$(b, c) \in R_1 \cup R_2 \cup (A_1 \times A_2)$.

Case 1: Suppose $(a, b) \in R_1$. Then, both a and b are elements of A_1. Since $A_1 \cap A_2 = \emptyset$, neither a nor b are elements of A_2. Thus, in particular, $(b, c) \notin R_2$ and $(b, c) \notin (A_1 \times A_2)$. Thus, $(b, c) \in R_1 \cup R_2 \cup (A_1 \times A_2)$ implies that $(b, c) \in R_1$.

Since R_1 is a partial order and $(a,b) \in R_1$ and $(b,c) \in R_1$, $(a,c) \in R_1$. Since a, b and c were arbitrary, in this case,$R_1 \cup R_2 \cup (A_1 \times A_2)$ is transitive.

Case 2: Suppose $(a,b) \in R_2$. Then, both a and b are elements of A_2. Since $A_1 \cap A_2 = \emptyset$, neither a nor b are elements of A_1. Thus, in particular, $(b,c) \notin R_2$ and $(b,c) \notin (A_1 \times A_2)$. Thus, $(b,c) \in R_1 \cup R_2 \cup (A_1 \times A_2)$ implies that $(b,c) \in R_2$. Since R_2 is a partial order and $(a,b) \in R_2$ and $(b,c) \in R_2$, $(a,c) \in R_2$. Since a, b and c were arbitrary, in this case,$R_1 \cup R_2 \cup (A_1 \times A_2)$ is transitive.

Case 3: Suppose $(a,b) \in A_1 \times A_2$. Then, $a \in A_1$ and $b \in A_2$. But then (b,c) cannot be in $R_1 \cup R_2 \cup (A_1 \times A_2)$. Thus, in this case, $R_1 \cup R_2 \cup (A_1 \times A_2)$ is transitive.

Thus, in either of these three exhaustive cases,$R_1 \cup R_2 \cup (A_1 \times A_2)$ is transitive. Since $R_1 \cup R_2 \cup (A_1 \times A_2)$ is reflexive, anti-symmetric and transitive, it is a partial order. ☐

c.

It may happen that R_1 is a total order on A_1 and R_2 is a total order on A_2, that $A_1 \cap A_2 = \emptyset$, but $R_1 \cup R_2$ is not a total order on $A_1 \cup A_2$. For example, let $A = \{a\}$ and $B = \{b\}$, $a \neq b$ so $A_1 \cap A_2 = \emptyset$, $R_1 = \{(a,a)\}$, and $R_2 = \{(b,b)\}$. They are both, then, total orders. Then neither (a,b) nor (b,a) are elements of $R_1 \cup R_2$. So it is not a total order.

theorem 4. *Suppose that R_1 is a total order on A_1 and R_2 is a total order on A_2. Suppose $A_1 \cap A_2 = \emptyset$. Then, $R_1 \cup R_2 \cup (A_1 \times A_2)$ is a total order on $A_1 \cup A_2$.*

Proof. Suppose that R_1 is a total order on A_1 and R_2 is a total order on A_2. Suppose $A_1 \cap A_2 = \emptyset$. We have already seen that $R_1 \cup R_2$ is a partial order. Let a and b be arbitrary. Suppose $a \in A_1 \cup A_2$ and $b \in A_1 \cup A_2$. Suppose $(a,b) \notin R_1 \cup R_2 \cup (A_1 \times A_2)$. Then $(a,b) \notin R_1$ and $(a,b) \notin R_2$ and $(a,b) \notin (A_1 \times A_2)$. We consider 4 cases:

Case 1: Suppose $a \in A_1$ and $b \in A_1$. Then, since $(a,b) \notin R_1$, and since R_1 is a total order, $(b,a) \in R_1$, thus $(b,a) \in R_1 \cup R_2 \cup (A_1 \times A_2)$. Since a and b were arbitrary, in this case $R_1 \cup R_2 \cup (A_1 \times A_2)$ is a total order.

Case 2: Suppose $a \in A_1$ and $b \in A_2$. Then, in this case, $(a,b) \in A_1 \times A_2$, contradicting our supposition that $(a,b) \notin (A_1 \times A_2)$

Case 3: Suppose $a \in A_2$ and $b \in A_1$. Then, $(b,a) \in A_1 \times A_2$, thus $(b,a) \in R_1 \cup R_2 \cup (A_1 \times A_2)$. Since a and b were arbitrary, in this case, $R_1 \cup R_2 \cup (A_1 \times A_2)$ is a total order.

Case 4: Suppose $a \in A_2$ and $b \in A_2$. Then, since $(a,b) \notin R_2$, and since R_2 is a total order, $(b,a) \in R_2$, thus $(b,a) \in R_1 \cup R_2 \cup (A_1 \times A_2)$. Since a and b were arbitrary, in this case $R_1 \cup R_2 \cup (A_1 \times A_2)$ is a total order.

Since $R_1 \cup R_2 \cup (A_1 \times A_2)$ is a total order in each of these 4 exhaustive cases, we have shown that $R_1 \cup R_2 \cup (A_1 \times A_2)$ is a total order. ☐

20.

theorem 5. *Suppose R is a partial order on a set A. Suppose $B \subseteq A$. Suppose $b \in B$ is the smallest element of B. Then, b is the greatest lower bound of B.*

Proof. Suppose R is a partial order on a set A. Suppose $B \subseteq A$. Suppose $b \in B$ is the smallest element of B. Let $x \in B$ be arbitrary. Since b is the smallest element of B, $(x, b) \in R$. Suppose z is a lower bound for B and $z \neq b$. Suppose $(b, z) \in R$. Then z is not a lower bound for B, a contradiction. Thus, $(b, z) \notin R$ and so b is the greatest lower bound. ☐

theorem 6. *Suppose R is a partial order on a set A. Suppose $B \subseteq A$. Suppose $b \in B$ is the largest element of B. Then, b is the least upper bound of B.*

Proof. Suppose R is a partial order on a set A. Suppose $B \subseteq A$. Suppose $b \in B$ is the largest element of B. Let $x \in B$ be arbitrary. Since b is the largest element of B, $(x, b) \in R$. Suppose z is an upper bound for B and $z \neq x$. Suppose $(z, b) \in R$. Then z is not a greater bound for B, a contradiction. Thus, $(z, b) \notin R$ and so b is the least upper bound. ☐