HW §4.3 Numbers 4,9,11,14,18,22

4.

a.

$$\{(a,c),(c,c),(d,a),(b,d),(d,b)\}$$

It is not reflexive, not symmetric, and not transitive.

b.

$$\{(a,b),(b,a),(a,d),(b,d)\}$$

It is not reflexive, not symmetric, and not transitive.

c.

$$\{(a,a),(b,b),(c,c),(d,d),(b,d),(d,b)\}$$

It is reflexive, symmetric, and transitive.

d.

$$\{(a,b),(a,c),(a,d),(b,d),(c,d)\}$$

It is not reflexive, not symmetric, but is transitive.

9.

a.

theorem 1. *Suppose A and B are two sets. Then, for every relation R from A to B, $R \circ i_A = R$.*

Proof. Suppose A and B are two sets. Let R be an arbitrary relation from A to B. Let $(x,y) \in R \circ i_A$ be arbitrary. Then, we can find $a \in A$ so that $(x,a) \in i_A$ and $(a,y) \in R$. Then $x = a$ by definition of i_A. Thus, $(x,y) \in R$. Since (x,y) was arbitrary, $R \circ i_A \subseteq R$

Let $(x,y) \in R$. Then, $(x,x) \in i_A$. Thus, $(x,y) \in R \circ i_A$. Since (x,y) was arbitrary, $R \circ i_A \supseteq R$. Since $R \circ i_A \subseteq R$ and $R \circ i_A \supseteq R$, $R \circ i_A = R$. □

b.

theorem 2. *Suppose A and B are two sets. Then, for every relation R from A to B, $R \circ i_A = R$.*

Proof. Suppose A and B are two sets. Let R be an arbitrary relation from A to B. Let $(x,y) \in R \circ i_A$ be arbitrary. Then, we can find $a \in A$ so that $(x,a) \in i_A$ and $(a,y) \in R$. Then $x = a$ by definition of i_A. Thus, $(x,y) \in R$. Since (x,y) was arbitrary, $R \circ i_A \subseteq R$

Let $(x,y) \in R$. Then, $(x,x) \in i_A$. Thus, $(x,y) \in R \circ i_A$. Since (x,y) was arbitrary, $R \circ i_A \supseteq R$. Since $R \circ i_A \subseteq R$ and $R \circ i_A \supseteq R$, $R \circ i_A = R$. □

11.

theorem 3. *Suppose R is a relation on A. Suppose R is reflexive. Then, $R \subseteq R \circ R$.*

Proof. Suppose R is a relation on A. Suppose R is reflexive. Let $(x, y) \in R$ be arbitrary. Since R is reflexive, $(x, x) \in R$. Thus, $(x, y) \in R \circ R$. □

14.

a.

theorem 4. *Suppose R_1 and R_2 are relations on A. Suppose R_1 and R_2 are reflexive. Then $R_1 \cap R_2$ is reflexive.*

Proof. Suppose R_1 and R_2 are relations on A. Suppose R_1 and R_2 are reflexive. Let $a \in A$ be arbitrary. Then, since R_1 is reflexive, $(a, a) \in R_1$. Then, since R_2 is reflexive, $(a, a) \in R_2$. Thus,$(a, a) \in R_1 \cap R_2$. Since a was arbitrary, $R_1 \cap R_2$ is reflexive. □

b.

theorem 5. *Suppose R_1 and R_2 are relations on A. Suppose R_1 and R_2 are symmetric. Then $R_1 \cap R_2$ is symmetric.*

Proof. Suppose R_1 and R_2 are relations on A. Suppose R_1 and R_2 are symmetric. Let $a \in A$ be arbitrary. Let $b \in A$ be arbitrary. Suppose $(a, b) \in R_1 \cap R_2$. Then, $(a, b) \in R_1$ and $(a, b) \in R_2$. Then, since R_1 is symmetric, $(b, a) \in R_1$. Then, since R_2 is symmetric, $(b, a) \in R_2$. Thus,$(b, a) \in R_1 \cap R_2$. Since a and b were arbitrary, $R_1 \cap R_2$ is symmetric. □

c.

theorem 6. *Suppose R_1 and R_2 are relations on A. Suppose R_1 and R_2 are transitive. Then $R_1 \cap R_2$ is transitive.*

Proof. Suppose R_1 and R_2 are relations on A. Suppose R_1 and R_2 are transitive. Let $a \in A$ be arbitrary. Let $b \in A$ be arbitrary. Let $c \in A$ be arbitrary. Suppose $(a, b) \in R_1 \cap R_2$. Suppose $(b, c) \in R_1 \cap R_2$. Then, $(a, b) \in R_1$ and $(a, b) \in R_2$. Then, $(b, c) \in R_1$ and $(b, c) \in R_2$. Then, since R_1 is transitive, $(a, c) \in R_1$. Then, since R_2 is transitive, $(a, c) \in R_2$. Thus,$(a, c) \in R_1 \cap R_2$. Since a and b and c were arbitrary, $R_1 \cap R_2$ is transitive. □

18.

theorem 7. *Suppose R and S are transitive relations on A. Suppose that $S \circ R \subseteq R \circ S$. Then, $R \circ S$ is transitive.*

Proof. Suppsoe R and S are transitive relations on A. Suppose that $S \circ R \subseteq R \circ S$. Let $a \in A$ be arbitrary. Let $b \in A$ be arbitrary. Let $c \in A$ be arbitrary. Suppose $(a, b) \in R \circ S$. Suppose $(b, c) \in R \circ S$. Then, we can find $d \in A$ so that $(a, d) \in S$ and $(d, b) \in R$. Then, we can find $e \in A$ so that $(b, e) \in S$ and $(e, c) \in R$. Thus, $(d, e) \in S \circ R$. Since $S \circ R \subseteq R \circ S$, $(d, e) \in R \circ S$. Thus, we can find $f \in A$ so that $(d, f) \in S$ and $(f, e) \in R$. Since S is transitive, $(a, f) \in S$. Since R is transitive, $(f, c) \in R$. Thus, $(a, c) \in R \circ S$. Since a, b and c were arbitrary, $R \circ S$ is transitive. □

22.

The proof is incorrect, as is the theorem. Let $A = \{x, y\}$ and $R = \emptyset$. Then, R is symmetric and transitive, but not reflexive.