HW §4.2 Numbers 2,5,8,12

2.
a.
The domain is the set of all persons who are male and who have a sibling. The range is the set of all persons who have a male sibling.
b.
The domain is $\mathbb{R}$. The range is $[-1, 1]$.

5.
a. $S^{-1} \circ R = \{(1,4), (3,5), (3,4)\}$
b. $R^{-1} \circ S = \{(4,1), (5,3), (4,3)\}$

8.
a.

theorem 1. *Suppose R is a relation from A to B and S is a relation from B to C. Then Dom$(S \circ R)$ $\subseteq$Dom(R).*

Proof. Suppose R is a relation from A to B and S is a relation from B to C. Let $a \in A$ be arbitrary. Suppose $a \in$Dom$(S \circ R)$. Then, we can find $b \in B$ and $c \in C$ so that $(a, b) \in R$ and $(b, c) \in S$. Thus, since $(a, b) \in R$, $a \in$Dom(R). □

b.

theorem 2. *Suppose R is a relation from A to B and S is a relation from B to C. Suppose Ran(R) $\subseteq$Dom(S). Then, Dom$(S \circ R)$ =Dom(R).*

Proof. Suppose R is a relation from A to B and S is a relation from B to C. Let $a \in A$ be arbitrary. Suppose Ran(R) $\subseteq$Dom(S). Let a be arbitrary. Suppose $a \in$ Dom$(S \circ R)$. Then, we have already seen in part a. that $a \in$Dom(R). Thus, Dom$(S \circ R)$ $\subseteq$Dom(R).
Let $a \in A$ be arbitrary. Suppose $a \in$Dom(R). Then, we can find $b \in B$ so that $(a, b) \in R$. This $b \in$Ran(R). Thus, $b \in$Dom(S), since Ran(R) $\subseteq$Dom(S). Since $b \in$Dom(S), we can find $c \in C$ so that $(b, c) \in S$. Thus, $(a, c) \in S \circ R$. Thus, $a \in$Dom$(S \circ R)$. Since a was arbitrary, *Dom*(R)$\subseteq$Dom$(S \circ R)$. Since *Dom*(R)$\subseteq$Dom$(S \circ R)$ and Dom$(S \circ R)$ $\subseteq$Dom(R) we have Dom$(S \circ R)$ =Dom(R). □

c.

theorem 3. *Suppose R is a relation from A to B and S is a relation from B to C. Then Ran$(S \circ R)$ $\subseteq$Ran(S).*

Proof. Suppose R is a relation from A to B and S is a relation from B to C. Let $c \in C$ be arbitrary. Suppose $c \in$Ran$(S \circ R)$. Then, we can find $b \in B$ and $c \in C$ so that $(a, b) \in R$ and $(b, c) \in S$. Thus, since $(b, c) \in S$, $c \in$Ran(S). □

theorem 4. *Suppose R is a relation from A to B and S is a relation from B to C. Suppose Dom(S) $\subseteq$Ran(R). Then, Ran$(S \circ R)$ =Ran(S).*

Proof. Suppose R is a relation from A to B and S is a relation from B to C. Suppose Dom(S) $\subseteq$Ran(R). Let c be arbitrary. Suppose $c \in$ Ran($S \circ R$). Then, we have already seen above that $c \in$Ran(S). Thus, Ran($S \circ R$) $\subseteq$Ran(S).
Let $c \in C$ be arbitrary. Suppose $c \in$Ran(S). Then, we can find $b \in B$ so that $(b, c) \in S$. Thus, $b \in$Dom(S). Thus, $b \in$Ran(R), since Dom(S) $\subseteq$Ran(R). Since $b \in$Ran(R), we can find $a \in A$ so that $(a, b) \in R$. Thus, $(a, c) \in S \circ R$. Thus, $c \in$Ran($S \circ R$). Since c was arbitrary, Ran(S) $\subseteq$Ran($S \circ R$). Since Ran(S)$\subseteq$Ran($S \circ R$) and Ran($S \circ R$) $\subseteq$Ran(S) we have Ran($S \circ R$) =Ran(S). □

12.
a. True.

theorem 5. *Suppose R is a relation from A to B and S and T are relations from B to C. Suppose $S \subseteq T$. Then, $S \circ R \subseteq T \circ R$.*

Proof. Suppose R is a relation from A to B and S and T are relations from B to C. Suppose $S \subseteq T$. Let a and c be arbitrary and suppose $(a, c) \in S \circ R$. Then, we can find $b \in B$ so that $(a, b) \in R$ and so that $(b, c) \in S$. Since $S \subseteq T$, $(b, c) \in T$. Since $(a, b) \in R$and $(b, c) \in T$, $(a, c) \in T \circ R$. Since a and c were arbitrary, $S \circ R \subseteq T \circ R$. □

b. True.

theorem 6. *Suppose R is a relation from A to B and S and T are relations from B to C. Then, $S \cap T \subseteq S \circ R \cap T \circ R$.*

Proof. Suppose R is a relation from A to B and S and T are relations from B to C. Let a and c be arbitrary and suppose $(a, c) \in S \cap T \circ R$. Then, we can find $b \in B$ so that $(a, b) \in R$ and so that $(b, c) \in S \cap T$. Thus, $(b, c) \in S$ and $(b, c) \in T$. Since $(a, b) \in R$, $(a, c) \in S \circ R$ and $(a, c) \in T \circ R$. Thus, $(a, c) \in S \circ R \cap T \circ R$. Since a and c were arbitrary, $S \cap T \subseteq S \circ R \cap T \circ R$. □

c. False. Let $A = \{a\}$, $B = \{b, b'\}$ and $C = \{c\}$. Let $R = \{(a, b), (a, b')\}$, $S = \{(b, c)\}$, and $T = \{(b', c)\}$. Then, $S \cap T = \emptyset$, so $S \cap T \circ R = \emptyset$ while $S \circ R \cap T \circ R = \{(a, c)\}$.
d. True.

theorem 7. *Suppose R is a relation from A to B and S and T are relations from B to C. Then, $S \cup T \circ R = S \circ R \cup T \circ R$.*

Proof. Suppose R is a relation from A to B and S and T are relations from B to C. Let a and c be arbitrary. Suppose $(a, c) \in S \cup T \circ R$. Then we can find $b \in S \cup T$ so that $(a, b) \in R$ and $(b, c) \in S \cup T$.
<u>Case 1:</u> Suppose $(b, c) \in S$. Then, $(a, c) \in S \circ R$. Thus, $(a, c) \in S \circ R \cup T \circ R$. Since a and c were arbitrary, $S \cup T \circ R \subseteq S \circ R \cup T \circ R$.
<u>Case 2:</u> Suppose $(b, c) \in T$. Then, $(a, c) \in T \circ R$. Thus, $(a, c) \in S \circ R \cup T \circ R$. Since a and c were arbitrary, $S \cup T \circ R \subseteq S \circ R \cup T \circ R$.
Let a and c be arbitrary. Suppose $(a, c) \in S \circ R \cup T \circ R$. Then either $(a, c) \in S \circ R$ or $(a, c) \in T \circ R$.

Case 1: Suppose $(a, c) \in S \circ R$. Then, we can find $b \in B$ so that $(a, b) \in R$ and $(b, c) \in S$. Since $(b, c) \in S$, $(b, c) \in S \cup T$. Thus, $(a, c) \in S \cup T \circ R$. Since a and c were arbitrary, $S \circ R \cup T \circ R \subseteq S \cup T \circ R$.
Case 2: Suppose $(a, c) \in T \circ R$. Then, we can find $b \in B$ so that $(a, b) \in R$ and $(b, c) \in T$. Since $(b, c) \in T$, $(b, c) \in S \cup T$. Thus, $(a, c) \in S \cup T \circ R$. Since a and c were arbitrary, $S \circ R \cup T \circ R \subseteq S \cup T \circ R$.
Thus, we have that $S \circ R \cup T \circ R \subseteq S \cup T \circ R$ since it is so in either of the two exhaustive cases.
Thus, since both $S \circ R \cup T \circ R \subseteq S \cup T \circ R$ and $S \cup T \circ R \subseteq S \circ R \cup T \circ R$, $S \cup T \circ R = S \circ R \cup T \circ R$. $\square$