

HW §3.6 Numbers 2,7,8,10

2.

**theorem 1.** For every  $x \in \mathbb{R}$  so that  $x \neq 0$  and  $x \neq 1$ , there exists a unique real number  $y$  such that  $y/x = y - x$ .

*Proof.* Let  $x \in \mathbb{R}$  be arbitrary. Let  $y = x^2/(x - 1)$ . Then,

$$\begin{aligned} y - x &= \frac{x^2}{x - 1} - x \\ &= \frac{x}{x - 1} \\ &= \frac{y}{x} \end{aligned}$$

□

7. Let  $U$  be any set.

a.

**theorem 2.** There exists a unique  $A \in \mathcal{P}(U)$ , such that for every  $B \in \mathcal{P}(U)$ ,  $A \cap B = B$ .

*Proof.* Let  $A = U$ . Then,  $A \in \mathcal{P}(U)$ . Let  $B \in \mathcal{P}(U)$  be arbitrary. Let  $x \in A \cap B = U \cap B$ . Then,  $x \in B$ . Thus,  $U \cap B \subseteq B$ .

Suppose  $x \in B$ . Then, since  $B \subseteq U$ ,  $x \in U$ . Thus,  $x \in U \cap B$ . Since  $x$  was arbitrary,  $B \subseteq U \cap B$ .

Thus,  $A \cap B = B$ .

Suppose  $C \in \mathcal{P}(U)$  has the property that for every  $B \in \mathcal{P}(U)$ ,  $C \cap B = B$ . Then, in particular,  $C \cap U = U$ . Since  $C \in \mathcal{P}(U)$ ,  $C \subseteq U$ . Let  $x \in U$ . Then,  $x \in C \cap U$ , thus  $x \in C$ . Thus,  $C = U$ . Thus,  $A = U$  is unique. □

b.

**theorem 3.** There exists a unique  $A \in \mathcal{P}(U)$ , such that for every  $B \in \mathcal{P}(U)$ ,  $A \cup B = B$ .

*Proof.* Let  $A = \emptyset$ . Then,  $A \in \mathcal{P}(U)$ . Let  $B \in \mathcal{P}(U)$  be arbitrary. Let  $x \in A \cup B = \emptyset \cup B$ . Then, since  $x \notin \emptyset$ ,  $x \in B$ . Thus,  $U \cup B \subseteq B$ .

Suppose  $x \in B$ . Then,  $x \in \emptyset \cup B$ . Since  $x$  was arbitrary,  $B \subseteq U \cup B$ .

Thus,  $A \cup B = B$ .

Suppose  $C \in \mathcal{P}(U)$  has the property that for every  $B \in \mathcal{P}(U)$ ,  $C \cup B = B$ . Then, in particular,  $C \cup \emptyset = \emptyset$ . Suppose  $x \in C$ . Then,  $x \in C \cup \emptyset$  so  $x \in \emptyset$ , a contradiction to the definition of  $\emptyset$ . Thus,  $x \notin C$ . Since  $x$  was arbitrary,  $C = \emptyset$ . Thus,  $A = \emptyset$  is unique. □

10.

**theorem 4.** *Suppose  $A$  is a set, and that for every family of sets  $\mathcal{F}$ , if  $\cup\mathcal{F} = A$ , then  $A \in \mathcal{F}$ . Then,  $A$  has a single element.*

*Proof.* Suppose  $A$  is a set, and that for every family of sets  $\mathcal{F}$ , if  $\cup\mathcal{F} = A$ , then  $A \in \mathcal{F}$ . Suppose  $A = \emptyset$ . Let  $\mathcal{F}$  be the empty collection. Then,  $\cup\mathcal{F} = A$ , but  $A \notin \mathcal{F}$ , which is a contradiction. Thus,  $A \neq \emptyset$ .

Now, let's show that  $A$  must have at least one element. To illustrate how this argument goes, let's show that  $A$  can't have two elements: Suppose  $A = \{x, y\}$ . Let  $\mathcal{F} = \{\{x\}, \{y\}\}$ . Then,  $\cup\mathcal{F} = A$ , but  $A \notin \mathcal{F}$ .

Now, let  $A = \{a_i\}_{i \in I}$ . where  $I$  has more than one element. Let  $\mathcal{F} = \{\{a_i\}\}_{i \in I}$ . Then,  $\cup\mathcal{F} = A$ , but  $A \notin \mathcal{F}$  since  $I$  has more than one element. Thus,  $A$  cannot have more than one element.  $\square$