

Solutions: §7.3. 6-20 even; 24-30 even

6. Let $x = 2 \tan \theta$. Then, $dx = 2 \sec^2 \theta d\theta$, so

$$\begin{aligned}\int_0^2 x^3 \sqrt{x^2 + 4} dx &= \int_0^{\pi/4} \tan^3 \theta \sqrt{4 \tan^2 \theta + 4} d\theta \\ &= \int_0^{\pi/4} \tan^3 \theta 2 \sec \theta d\theta \\ &= \int_0^{\pi/4} (\sec^2 \theta - 1) 2 \sec \theta \tan \theta d\theta\end{aligned}$$

Let $u = \sec \theta$. Then, $du = \sec \theta \tan \theta d\theta$. So,

$$\begin{aligned}&= \int_1^{\sqrt{2}} (u^2 - 1) du \\ &= \frac{1}{3} u^3 - u \Big|_1^{\sqrt{2}} \\ &= \sqrt{2}^3 - \sqrt{2}\end{aligned}$$

8. Let $x = a \sec \theta$. Then, $dx = a \sec \theta \tan \theta d\theta$, so

$$\begin{aligned}\int \frac{\sqrt{x^2 - a^2}}{x^4} dx &= \int \frac{\sqrt{a^2 \sec^2 \theta - a^2}}{\sec^4 \theta} \sec \theta \tan \theta d\theta \\ &= \int \frac{\tan^2 \theta}{\sec^3 \theta} d\theta \\ &= \int \frac{\sin^2 \theta}{\cos^2 \theta} \cos^3 \theta d\theta \\ &= \int \sin^2 \theta \cos \theta d\theta \\ &= \frac{1}{3} \sin^3 \theta + C \\ &= \frac{\sqrt{(x^2 - a^2)^3}}{3a^3}\end{aligned}$$

10. Let $t = \sqrt{2} \tan \theta$, then $dt = \sqrt{2} \sec^2 \theta$. So,

$$\begin{aligned}\int \frac{t^5}{\sqrt{t^2 + 2}} dt &= \int \frac{\sqrt{2}^5 \tan^5 \theta}{\sqrt{2 \tan^2 \theta + 2}} \sqrt{2} \sec^2 \theta \\ &= 4\sqrt{2} \int \frac{\tan^5 \theta}{\sec \theta} \sec^2 \theta d\theta \\ &= 4\sqrt{2} \int \frac{\sin^5 \theta}{\cos^4 \theta} d\theta\end{aligned}$$

$$\begin{aligned}
&= 4\sqrt{2} \int \frac{(1 - \cos^2 \theta)^2}{\cos^4 \theta} \sin \theta d\theta \\
&= 4\sqrt{2} \int (\sec^4 \theta - 2 \sec^2 \theta + 1) \sin \theta d\theta \\
&= 4\sqrt{2} \int \tan \theta \sec^3 \theta - \tan \theta \sec \theta + \sin \theta d\theta \\
&= 4\sqrt{2} \left(\frac{1}{3} \sec^3 \theta - \sec \theta - \cos \theta \right) + C \\
&= 4\sqrt{2} \left(\frac{\sqrt{(t^2 + 2)^3}}{3\sqrt{8}} - \sqrt{\frac{t^2 + 2}{2}} - \sqrt{\frac{2}{t^2 + 2}} \right)
\end{aligned}$$

12. Let $x = 2 \tan \theta$. Then, $dx = 2 \sec^2 \theta d\theta$. So,

$$\begin{aligned}
\int_0^1 x \sqrt{x^2 + 4} &= \int_0^{\pi/6} 2 \tan \theta \sec^3 \theta d\theta \\
&= \int_0^{\pi/6} 2 \tan \theta \sec^3 \theta d\theta \\
&= \frac{1}{3} \sec^3 \theta \Big|_0^{\pi/6} \\
&= \frac{1}{3} \left(\frac{\sqrt{27}}{8} - 1 \right)
\end{aligned}$$

14. Let $u = \sqrt{5} \sin \theta$. Then, $du = \sqrt{5} \cos \theta d\theta$. So,

$$\begin{aligned}
&= \int \frac{du}{u\sqrt{5-u^2}} = \frac{\sqrt{5} \cos \theta d\theta}{\sqrt{5} \sin \theta \sqrt{5-5\sin^2 \theta}} \\
&= \int \frac{\sqrt{5} \cos \theta d\theta}{5 \sin \theta \cos \theta} \\
&= \frac{\sqrt{5}}{5} \ln |\csc \theta - \cot \theta| + C \\
&= \frac{\sqrt{5}}{5} \ln \left| \sqrt{\frac{5}{5-u^2}} - \frac{\sqrt{5-u^2}}{u} \right|
\end{aligned}$$

16. Let $x = 3/4 \sec \theta$. Then, $dx = \frac{3}{4} \sec \theta \tan \theta$. So,

$$\begin{aligned}
&= \int \frac{dx}{x^2 \sqrt{16x^2 - 9}} = \int \frac{\frac{3}{4} \sec \theta \tan \theta}{\frac{9}{16} \sec^2 \theta \sqrt{9 \sec^2 \theta - 9}} \\
&= \int \frac{\frac{3}{4} \sec \theta \tan \theta}{\frac{27}{16} \sec^2 \theta \tan \theta} \\
&= \frac{1}{3} \int \cos \theta d\theta
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \sin \theta + C \\
&= \frac{\sqrt{16x^2 - 9}}{12x} + C
\end{aligned}$$

18. Let $x = \frac{b}{a} \sec \theta$. Then, $dx = \frac{b}{a} \sec \theta \tan \theta$. So,

$$\begin{aligned}
\int \frac{dx}{[(ax)^2 - b^2]^{3/2}} &= \int \frac{\frac{b}{a} \sec \theta \tan \theta}{[(a\frac{b}{a} \sec^2 \theta) - b^2]^{3/2}} \\
&= \int \frac{\frac{b}{a} \sec \theta \tan \theta d\theta}{[b \tan \theta]^3} \\
&= \int \frac{\cos \theta d\theta}{ab^2 \sin^2 \theta}
\end{aligned}$$

Letting $u = \sin \theta$ and un-substituting, we get

$$= \frac{-1}{ab^2} \frac{\sqrt{(ax)^2 - b^2}}{4x} + C$$

20. Let $t = 5 \sin \theta$. Then, $dt = 5 \cos \theta d\theta$. So,

$$\begin{aligned}
\int \frac{t}{\sqrt{25 - t^2}} dt &= \int \frac{5 \cos \theta d\theta}{\sqrt{25 - 25 \sin^2 \theta}} \\
&= \int d\theta \\
&= \theta \\
&= \sin^{-1}\left(\frac{t}{5}\right)
\end{aligned}$$

24.

$$\int \frac{dt}{\sqrt{t^2 - 6t + 13}} = \int \frac{dt}{\sqrt{(t-3)^2 + 4}}$$

Let $t - 3 = 2 \tan \theta$. Then, $dt = 2 \sec^2 \theta d\theta$. Which gives

$$\begin{aligned}
&= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4 \tan^2 \theta + 4}} \\
&= \int \sec \theta d\theta \\
&= \ln |\sec \theta + \tan \theta| + C \\
&= \ln \left| \frac{\sqrt{t^2 - 6t + 13}}{2} + \frac{t-3}{2} \right| + C
\end{aligned}$$

26.

$$= \int \frac{x^2}{\sqrt{4x - x^2}} dx = \int \frac{x^2}{\sqrt{4 - (2-x)^2}} dx$$

Let $2 - x = 2 \cos \theta$, then $dx = -2 \sin \theta$. So,

$$\begin{aligned} &= \int \frac{4 \cos^2 \theta}{\sqrt{4 - 4 \cos^2 \theta}} 2 \sin \theta \\ &= \int 2 \cos^2 \theta d\theta \\ &= \theta + \frac{1}{2} \sin \theta + C \\ &= \cos^{-1}\left(\frac{2-x}{2}\right) + \frac{1}{4} \sqrt{4x - x^2} + C \end{aligned}$$

28.

$$= \int \frac{dx}{(5 - 4x - x^2)^{5/2}} = \int \frac{dx}{(1 - (2-x)^2)^{5/2}}$$

Let $2 - x = \cos \theta$. Then, $dx = -\sin \theta$. So we get

$$\begin{aligned} &= \int \frac{\sin \theta d\theta}{(1 - \cos^2 \theta)^{5/2}} \\ &= \int \csc^4 \theta d\theta \\ &= \int (1 + \cot \theta) \csc^2 \theta d\theta \\ &= -\cot \theta - \frac{1}{2} \cot^2 \theta + C \\ &= \frac{2-x}{\sqrt{-3+4x-x^2}} - \frac{1}{2} \frac{(2-x)^2}{-3+4x-x^2} + C \end{aligned}$$