HW §11.8 Numbers 4,8,12,16,26,28

4.

$$\lim_{n\to\infty}\left|\frac{x^{n+1}\sqrt{n+1}}{x^n\sqrt{n}}\right| = |x|$$

Thus, the series

$$\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$$

is absolutely covergent if $|x| < 1$ and divergent if $|x| > 1$. Now, if $x = 1$ we have

$$\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

which diverges by the integral test. If $x = -1$ we have

$$\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

which converges by the integral test. Thus, the radius of convergence is 1 and the interval $[-1, 1)$.

8.

$$\lim_{n\to\infty}\left|\sqrt[n]{n^n x^n}\right|$$

$$= \lim_{n\to\infty}|nx|$$

which is ∞ if $x \neq 0$ and 0 if $x = 0$. Thus, the series

$$\sum_{n=1}^{\infty} n^n x^n$$

converges if $x = 0$ and diverges otherwise. Thus, the radius of convergence is 0 and the interval $\{0\}$.

12.

$$\lim_{n\to\infty}\left|\frac{x^{n+1}5^n n^5}{x^n 5^{n+1}(n+1)^5}\right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{5} \left(\frac{n}{n+1} \right)^5 \right|$$
$$= \left| \frac{x}{5} \right|$$

Thus, the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$$

is absolutely covergent if $|x| < 5$ and divergent if $|x| > 5$. Now, if $x = 5$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$$

$$= \sum_{n=1}^{\infty} \frac{5^n}{5^n n^5}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^5}$$

which converges by the integral test. Thus so too does

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$$

since absolute convergence implies convergence. Thus, the radius of convergence is 1 and the interval $[-5, 5]$.

16.
Since

$$\lim_{n\to\infty} \left| \frac{(n+1)^3 (x-5)^{n+1}}{n^3 (x-5)^n} \right|$$
$$= |(x-5)|$$

Thus, the series

$$\sum_{n=1}^{\infty} n^3 (x-5)^n$$

converges absolutely when $|x-5| < 1$, that is when $4 < x < 6$. It diverges when $|x-5| > 1$. Now, when $x = 4$ we have the series

$$\sum_{n=1}^{\infty} n^3 (-1)^n$$

which diverges by the test for divergence ($\lim -1^n n^3 \neq 0$). Similarly, when $x = 6$ we have the series

$$\sum_{n=1}^{\infty} n^3$$

which diverges again by the test for divergence. So the radius of convergence is 1 and the interval $(4, 6)$.

26.

$$\lim_{n\to\infty}\left|\frac{(2x+3)^{n+1}(n\ln n)}{(2x+3)^n(n+1)\ln(n+1)}\right|$$
$$= |2x+3|$$

Thus. the series is absolutely convergent if $|2x+3| < 1$ and divergent if $|2x+3| > 1$ Suppose $2x+3 = -1$. Then,

$$\sum_{n=1}^{\infty}(-1)^n\frac{(2x+3)^n}{n\ln n}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n\ln n}$$

which is divergent by the integral test. On the other hand, suppose $2x+3 = 1$. Then,

$$\sum_{n=1}^{\infty}(-1)^n\frac{(2x+3)^n}{n\ln n}$$
$$= \sum_{n=1}^{\infty}(-1)^n\frac{1}{n\ln n}$$

which converges by the alternating series test. Thus, the radius of convergence is $1/2$ and the interval is $(-2, -1]$.

28.
Since

$$\lim_{n\to\infty}\left|\frac{[2\cdot 4\cdot 6\cdots(2n+2)][1\cdot 3\cdot 5\cdots(2n-1)]x^{n+1}}{[1\cdot 3\cdot 5\cdots(2n+1)][2\cdot 4\cdot 6\cdots(2n)]x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{2n+2}{2n+1}x\right|$$
$$= |x|$$

we can say that

$$\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}x^n$$

converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Now, if $x = 1$, then we have the series

$$\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}$$

which diverges by the test for divergence since the sequence

$$\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

is clearly positive for all n and increasing. Similarly, when $x = -1$ we get the series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} (-1)^n$$

which diverges since

$$\lim_{n \to \infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} (-1)^n \neq 0$$

Thus, the radius of convergence is 1 and the interval is $(-1, 1)$.